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THE CONVERGENCE OF PERIODIC WAVES TO SOLITARY WAVES IN THE LONG--ETC(U)
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THE CONVERGENCE OF PERIODIC WAVES TO SOLITARY WAVES IN THE LONG WAVE LIMIT

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ABSTRACT

It is shown that large amplitude solitary water-waves arise as the limit of periodic waves whose wavelength increases indefinitely. This result is obtained after a new version of the Nekrasov integral equation for periodic waves has been derived. Its resemblance to the equation for solitary waves [1] leads to this convergence result once the global existence proof for solitary waves given in [1] has been taken into account.

AMS (MOS) Subject Classification: 76.45, 45G05, 45C05, 47H15.

Key Words: Nekrasov's integral equation; asymptotic convergence; long-wave limit.

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

Now that large amplitude solitary waves have been proved to exist [1], it is natural to consider whether periodic water-waves converge to solitary waves as the wavelength increases indefinitely. Numerical evidence suggests that this is the case, and the substance of this paper is to provide a proof of this observation, and to elucidate the precise circumstances under which this claim is valid. Previous results along these lines were proved under the assumption that the amplitudes were small; no such assumption is necessary here.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE CONVERGENCE OF PERIODIC WAVES TO SOLITARY WAVES IN THE LONG-WAVE LIMIT

J. F. Toland

INTRODUCTION

1.1. Introductory remarks

Under consideration are the steady two-dimensional waves which can arise as the free surface of a heavy, ideal liquid acted on by gravity, and contained in a channel of infinite extent with a horizontal bottom, in the absence of surface tension effects. It is well known that both periodic waves [8], [10] and solitary waves [1] of large amplitude may occur in these circumstances. A precise account of the free boundary-value problem presented by this situation is given in the next section, and various physical parameters describing the flows are introduced. After some basic results about conformal mappings and Jacobi elliptic functions have been recorded in section 1.3, the method of Nekrasov [16] is used to reduce the existence questions for the free boundary-value problems to a similar question for nonlinear integral equations. While the solitary wave equation (1.47) is the familiar one (a precise account of which is given in [1]), the equation for periodic waves (1.31) is new in the form given here. It is, of course, equivalent to the usual integral equation for periodic waves [8], [10], [20] but it has distinct advantages for our purposes. First of all, the domain of the independent variable is $[-\pi,\pi]$, and is independent of the wavelength of the waves being considered. Moreover its kernel is $\ell n|(\sin (s+t)/2)/(\sin (s-t)/2)|$ whose behaviour is well known. Most important of all is its striking resemblance to the approximation used in [1; section 3.2] to prove the existence of large amplitude solitary waves (a point we shall return to later). Throughout this section we emphasize the role which various physical

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parameters play in these integral equation formulations. For example, in the periodic case the wavelength and the mean depth are chosen *a priori* and appear as constants in the equation, whereas the mean velocity, the flux and the flow velocity at the crest depend on the solution of the equation being considered. An account of all this is given in Theorems 1.5 and 1.6.

After a few remarks in section 2.1 about recent developments in the theory of large amplitude periodic water-waves, section 2.2 is devoted to a summary and sketch of the proof of the global bifurcation theorem of periodic waves of wavelength λ on a flow of mean depth h, where λ and h are any given positive real numbers. It is shown that there exists a connected set of such waves containing waves of all amplitudes up to and including a wave of greatest height (of wavelength λ on a flow of mean depth h). This connected set contains a wave whose maximum angle of inclination to the horizontal is β, for any $\beta \in (0,\pi/6)$. Moreover the phase speeds of all such periodic waves is bounded away from zero and infinity. A comment on the relation between these results and those of [8], [10], is in order, and is to be found in section 2.1.

Finally, in section 2.3, the main result of this paper is proved, and is roughly speaking the following: if h is fixed, then as $\lambda \to \infty$ the connected sets of periodic waves of wavelength λ on a flow of mean depth h, converge in a certain sense to a connected set of solitary waves whose asymptotic height is h. This connected set enjoys all the properties of the connected set $\mathcal{C}$ mentioned in [1; Theorem 3.9], and the behaviour of the corresponding waves is described in [1; section 4]. The global existence of solitary waves is already known [1]; what is new here is that periodic waves converge to solitary waves in the long wave limit. An easy corollary of our general result in this direction is the following:

COROLLARY 2.6. *For each* β, $0 < \beta < \pi/6$, *and* $h, \lambda > 0$ *there exists on a flow of mean depth* h, *a periodic, symmetric water-wave of wavelength* λ, *the free surface of which subtends an angle* β *with the horizontal at its steepest point*. *If* h *is fixed, and* $\lambda_n \uparrow \infty$ *as* $n \to \infty$, *then a subsequence of the periodic-wave profiles converge uniformly on compact subsets of* $\mathbb{R}$ *to the profile of a steady solitary water-wave whose free surface subtends a maximum angle of* β *with the horizontal, and whose asymptotic depth is* h.

Such results as these may be regarded as global versions of the theorems of Ter-Krikorov [18], [19] and Lavrentiev [12], who proved the existence of small-amplitude solitary waves by proving the convergence of small-amplitude periodic waves to solitary waves as their wavelength increases indefinitely. (See [3] for a similar global treatment of a different problem.) It is worth noting that, because the mathematical theory of large amplitude water-waves lacks any form of global uniqueness result, we cannot claim that all solitary waves may be described as the long-wave limit of a sequence of periodic waves. The results of section 2.3 follow immediately by the methods of [1; section 3] once the similarity between (1.31) and equation (3.12) of [1] has been noted. The analysis presented here has the advantage that the linearization about the zero solution of (1.31) is well understood because exact solutions can be found. It may be regarded as a small step towards finding theoretical confirmation of the very striking numerical results given in [4].

1.2. *The water-wave problems*

The question being considered is the existence problem for steady, two dimensional waves on the surface of an ideal liquid acted on by gravity. In this section two possible types of flow are considered.

(a) *A symmetric, periodic flow of wavelength* λ *whose mean depth is* h. If such a flow exists and if the free surface has a unique maximum per wavelength

then a cross-section of the flow perpendicular to the wave crests may be identified with a region in the z-plane between the line $y = 0$ and a curve $\{x+iH_\lambda(x) : x \in \mathbb{R}\}$. Here $H_\lambda : \mathbb{R} \to (0,\infty)$ is a function of period λ which is even and is decreasing on the interval $(0,\lambda/2)$ (see figure 1). One wavelength of this flow then occupies the region S_λ bounded by the lines $x = \pm\lambda/2$, $y = 0$ and the free surface $\Gamma_\lambda = \{x+iH_\lambda(x) : x \in (-\lambda/2,\lambda/2)\}$. Since the flow is supposed to be incompressible and irrotational, there exists a complex potential $\omega = \phi + i\psi$, which is related to the velocity $(u(z),v(z))$ of the flow at a point $z \in S_\lambda$ by the expression

$$u(z) - iv(z) = -\frac{d\omega}{dz} = (-\phi_x,\phi_y) = (-\psi_y,-\psi_x) \quad . \tag{1.1}$$

Since the flow is symmetric about $x = 0$, ω must satisfy the relationship

$$\overline{\frac{d\omega}{dz}}(z) = \frac{d\omega}{dz}(-\bar{z}) \quad , \tag{1.2}$$

whence

$$\psi_x(z) = -\psi_x(-\bar{z}) \quad , \tag{1.3}$$

and

$$\psi(z) = \psi(-\bar{z}) \quad . \tag{1.4}$$

In particular ψ_x is zero on the imaginary axis and so, by periodicity,

$$\psi_x(z) = -\phi_y(z) = 0 \quad \text{if} \quad \text{Real } z = \pm\lambda/2 \quad . \tag{1.5}$$

Let $C = \{z(t) : t \in [0,1]\}$ be any simple curve in S_λ joining two points $\pm\lambda/2 + iy$. Then

$$\begin{aligned}\int_C u(z) + iv(z)dz &= \omega(-\lambda/2+iy) - \omega(\lambda/2+iy) \\ &= \phi(-\lambda/2+iy) - \phi(\lambda/2+iy) \quad ,\end{aligned}$$

by (1.4),

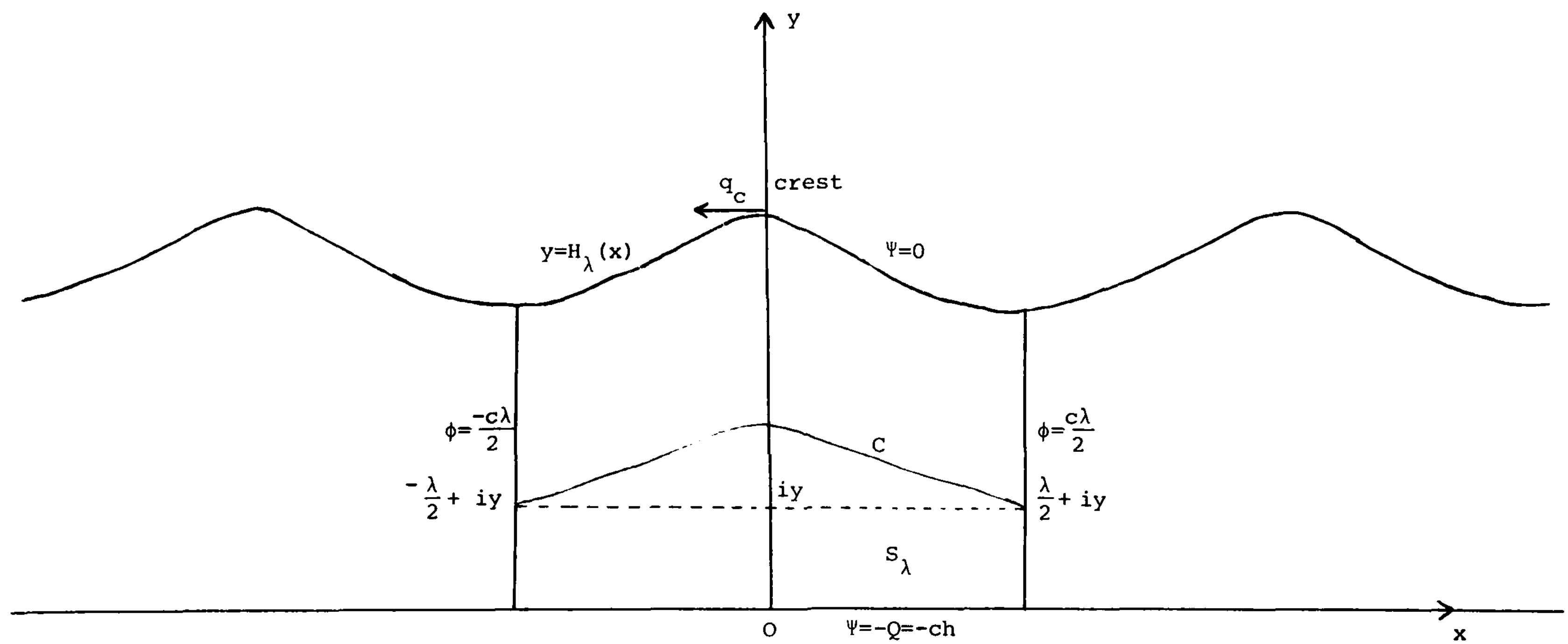

Figure 1. A steady periodic wave of wavelength λ on a flow of mean depth h. The region occupied by one wavelength is S_λ. The mean velocity of the flow is $-c$, and its speed at the wave crest is q_c.

$$= -(\phi(\lambda/2) - \phi(-\lambda/2))$$

by (1.5). Thus, for a given periodic flow,

$$\frac{1}{\lambda} \int_C u(z) + iv(z)dz = -(\phi(\lambda/2) - \phi(-\lambda/2))/\lambda \tag{1.6}$$

is called the mean velocity and is denoted by $-c$. (If the flow is considered in a frame of reference relative to which the mean velocity is zero, then c is the phase speed of the wave.) Since the bottom $(y = 0)$ and the free surface Γ_λ are streamlines, the stream-function ψ must be constant on both, and without loss of generality we may suppose that

$$\psi(z) = 0 \quad \text{if} \quad z \in \Gamma_\lambda \quad . \tag{1.7}$$

Since h is the mean depth of the flow,

$$\psi(z) = -Q, \quad \text{if} \quad \text{Imag } z = 0 \quad , \tag{1.8}$$

where

$$Q = ch \quad . \tag{1.9}$$

(Note that for a given flow the mean depth is not to be confused with an integral average of the height of the free surface. It is defined by (1.9) once the flux Q of the flow is known. By definition Q is the value of ψ on the bottom when ψ has been normalized so that $\psi = 0$ on the free surface.) Finally, since Γ is a free streamline, the pressure is a constant there, and Bernoulli's theorem then implies that

$$\frac{1}{2} |\nabla\phi(z)|^2 + g \text{ Imag } z = \text{constant} \tag{1.10}$$

for all $z \in \Gamma_\lambda$, where g is the acceleration due to gravity.

The existence question for this type of periodic flow is first one of finding the region S_λ occupied by one wavelength of the flow, and then one of finding ϕ and ψ such that a periodic flow of wavelength λ and mean depth h

occupies S_λ. It must be shown that ϕ and ψ satisfies all the conditions (1.1) - (1.5), (1.7), (1.8) and (1.10) in S_λ where Q is given by (1.9) and c is given by (1.6).

(b) Solitary waves on a flow of asymptotic depth h. By a steady solitary wave is meant a symmetric two-dimensional flow whose free surface is in the form of a single symmetric wave of elevation, whose extent is infinite, and which is asymptotic to a finite height at $\pm\infty$ (see Figure 2). The flow at $\pm\infty$ is supposed to be approximately uniform horizontal flow from right to left in the channel. The boundary-value problem posed by this situation is first to find the flow domain S bounded by the line $y = 0$ and a curve $\Gamma = \{x+iH(x) : x \in \mathbb{R}\}$ where the even function H is decreasing, and

$$\lim_{|x|\to\infty} H(x) = h \tag{1.11}$$

and then to find a complex potential ω satisfying all the boundary conditions, which in this case take the following form. The relationship between the complex potential and the velocity field is given by (1.1), and since the flow is symmetrical (1.2) must also be satisfied. Since the flow is supposed approximately uniform and horizontal at points of S far from the crest, there results that

$$\lim_{\substack{|z|\to\infty \\ z\in S}} u(z) - iv(z) = \lim_{\substack{|z|\to\infty \\ z\in S}} \frac{-d\omega}{dz}(z) = -c \quad , \tag{1.12}$$

where $-c$ is the asymptotic velocity of the steady flow. (In a frame of reference relative to which t he asymptotic speed is zero, c is the phase speed of the wave.) Since Γ and the bottom are both streamlines, we may suppose that

$$\psi = 0 \quad \text{on} \quad \Gamma \tag{1.13}$$

and

$$\psi = -ch \quad \text{if} \quad \text{Imag } z = 0 \quad . \tag{1.14}$$

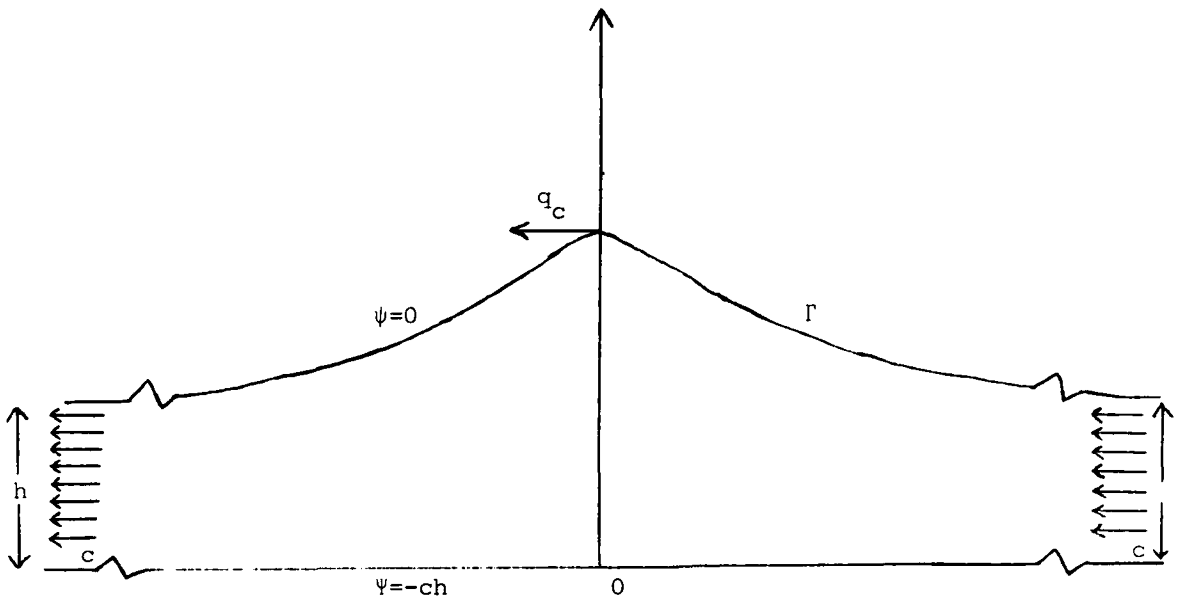

Figure 2. The region occupied by a steady solitary wave, of asymptotic velocity c (from right to left) and asymptotic height h.

The boundary condition (1.13) is a normalization, as before, and (1.14) follows from (1.11) because the stream function is a constant on the bottom, and by (1.12), $\lim_{\substack{|z|\to\infty \\ z\in S}} \psi_y(z) \to c$. Finally, since Γ is a <u>free</u> streamline, Bernoulli's theorem requires that (1.10) must hold on Γ.

If, by analogy with the periodic case, the mean velocity of a solitary wave is calculated using the formula $\lim_{\lambda\to\infty} \int_{-\lambda/2+iy}^{\lambda/2+iy} (u(z)+iv(z))dz$, for any $y \in (0,h)$, then it follows from (1.12) that this value coincides with the asymptotic velocity. Since the flux of the solitary wave is ch, it follows that the mean depth, and the asymptotic depth coincide.

It is appropriate at this stage to mention one further parameter associated with a flow, solitary or periodic. In either case q_c is used to denote the <u>velocity</u> of <u>the steady flow at the crest</u> of a wave.

As is well-known both the free boundary-value problems described above can be reformulated as nonlinear integral equations [15], [16]. In section 1.4 this

formulation is discussed in detail, but before we can do that we need to introduce some conformal mappings. The approach of the next two sections is suggested by the remarks in the appendix of [8].

1.3. Preliminary mapping theorems

A treatment of Jacobi's elliptic functions which is adequate for our purposes is given in [5]. Throughout the section and in all of the sequel, h is an arbitrary, but fixed, positive real number.

For $k \in (0,1)$, $\mathrm{sn}(\cdot,k)$ denotes the odd Jacobi elliptic function of modulus k with primative periods $4K$ and $2iK'$ and simple poles with residues $1/k$ or $-1/k$ at points congruent to iK' or to $2K + iK'$ (mod $4K$, $2iK'$) respectively. The primative periods are given in terms of k by the formulae

$$K = \int_0^1 (1 - x^2)^{-\frac{1}{2}} (1 - k^2x^2)^{-\frac{1}{2}} dx \quad , \tag{1.15}$$

and

$$K' = \int_0^1 (1-x^2)^{-\frac{1}{2}} (1 - (1-k^2)x^2)^{-\frac{1}{2}} dx \quad . \tag{1.16}$$

Clearly K and K' are monotone functions of $k \in (0,1)$, $K \downarrow \frac{\pi}{2}$ and $K' \uparrow \infty$ as $k \downarrow 0$, while $K \uparrow \infty$ and $K' \downarrow \frac{\pi}{2}$ as $k \uparrow 1$.

For any $\lambda > 0$ let k_λ be the modulus of the unique function $\mathrm{sn}(\cdot,k_\lambda)$ such that

$$\frac{\lambda}{h} = 4 \frac{K_\lambda}{K_\lambda'} \quad , \tag{1.17}$$

where K_λ and K_λ' are defined in terms of k_λ by (1.15), (1.16). Since h is fixed, k_λ, K_λ and K_λ' are monotone functions of λ and

$$\frac{2K_\lambda}{\lambda} \to \frac{\pi}{4h} \tag{1.18}$$

as $\lambda \to \infty$. For convenience with notation we will use s_λ to denote the elliptic function $\mathrm{sn}(\cdot,k_\lambda)$ for all $\lambda > 0$, and s_∞ to denote the analytic function tanh which is the pointwise limit of s_λ as $\lambda \to \infty$ [5; page 414, ex. 1].

It is well known [5; page 414, ex. 4] that the mapping $\tilde{p}_\lambda$ from the complex ζ- plane into the complex ξ-plane defined by

$$\tilde{p}_\lambda(\zeta) = -k_\lambda s_\lambda^2(2K_\lambda(\zeta + ih)/\lambda))$$

is a conformal mapping of the region $R_\lambda = \{\zeta = X+i\eta : -\lambda/2 < X < \lambda/2, -h < \eta < 0\}$ onto the region $\mathcal{D}' = \{\xi = re^{is} : 0 < r < 1, -\pi < s < \pi\}$, which maps the boundary portion $\Gamma_\lambda' = \{\zeta \in \bar{R}_\lambda : \zeta = X+i0, X \in [-\lambda/2,\lambda/2)\}$ onto the set $\{\xi = e^{is} : -\pi < s \le \pi\}$. Let $p_\lambda : [-\lambda/2,\lambda/2) \to (-\pi,\pi]$ be defined as follows:

$$p_\lambda(X) = s, \quad \text{for} \quad X \in [-\lambda/2,\lambda/2)$$

if and only if $s \in (-\pi,\pi]$ and

$$\tilde{p}_\lambda(X+i0) = e^{is} \quad .$$

Another, more convenient way of saying this is that for $X \in [-\lambda/2,\lambda/2)$, $p_\lambda(X) = s$ if and only if $s \in (-\pi/2,\pi/2]$ and

$$\cos\frac{s}{2} + i\sin\frac{s}{2} = -ik_\lambda^{1/2} s_\lambda(2K_\lambda(X+ih)/\lambda)$$

$$= -i\left\{\frac{(1+k_\lambda)s_\lambda(2K_\lambda X/\lambda)+ic_\lambda(2K_\lambda X/\lambda)d_\lambda(2K_\lambda X/\lambda)}{1+k_\lambda s_\lambda^2(2K_\lambda X/\lambda)}\right\} , \tag{1.19}$$

where c_λ and d_λ denote the even Jacobi elliptic functions $\mathrm{cn}(\cdot,k_\lambda)$ and $\mathrm{dn}(\cdot,k_\lambda)$. (Algebraic identities and rules for differentiating the functions c_λ, d_λ and s_λ and given in [5; page 384]. The expression (1.19) follows from the relation (1.17) and [5; page 396, eq. 3].) Let $\tilde{q}_\lambda : \mathcal{D}' \to R_\lambda$ denote the inverse

of $\tilde{p}_\lambda$, and let $q_\lambda : (-\pi,\pi] \to [-\lambda/2,\lambda/2)$ denote the inverse of p_λ. From equation (1.19) it follows that

$$\sin\frac{s}{2} = -\frac{(1+k_\lambda)s_\lambda(2K_\lambda q_\lambda(s)/\lambda)}{1+k_\lambda s_\lambda^2(2K_\lambda q(s)/\lambda)} \quad ,$$

which, upon differentiating with respect to s and using (1.19) along with the identitites in [5; page 384] yields

$$\frac{1}{2}\cos\frac{s}{2} = -\frac{2K_\lambda(1+k_\lambda)}{\lambda}\left\{\frac{1-k_\lambda s_\lambda^2(2K_\lambda q_\lambda(s)/\lambda)}{1+k_\lambda s_\lambda^2(2K_\lambda q_\lambda(s)/\lambda)}\right\}\cos\frac{s}{2}\, q'(s) \quad . \tag{1.20}$$

But, by the algebraic identities relating s_λ, c_λ and d_λ there results that

$$(1-k_\lambda s_\lambda^2)^2 = c_\lambda^2 d_\lambda^2 + (1-k_\lambda)^2 s_\lambda^2 \quad ,$$

and so (1.19) and (1.20) together yield the following expression for q_λ' :

$$q_\lambda'(s) = -(\lambda/4K_\lambda(1+k_\lambda))\left[\cos^2\frac{s}{2} + \left(\frac{1-k_\lambda}{1+k_\lambda}\right)^2 \sin^2 s\right]^{-1/2} \quad . \tag{1.21}$$

For convenience with notation we define the following expressions:

$$\left.\begin{aligned} f_\lambda(s) &= \frac{1}{2}\left[\cos^2\frac{s}{2} + \left(\frac{1-k_\lambda}{1+k_\lambda}\right)^2\sin^2 s\right]^{-1/2}, \\ f(s) &= \frac{1}{2}\sec\frac{s}{2} \quad , \end{aligned}\right\} \tag{1.22}$$

for all $s \in (-\pi,\pi)$, and

$$\Lambda = \lambda/2K_\lambda(1+k_\lambda) \quad . \tag{1.23}$$

Recall from (1.18) that

$$\Lambda \to 2h/\pi \quad \text{as} \quad \lambda \to \infty \quad . \tag{1.24}$$

Since the only zeros of $d\tilde{p}_\lambda/d\zeta$ occur at $\zeta = -ih$ and $\pm\lambda/2 - ih$ the real and imaginary parts of $\tilde{q}_\lambda$ satisfy the Cauchy-Riemann conditions on the boundary portion $\{e^{it} : -\pi < t < \pi\}$ of $\partial\mathcal{D}'$. Hence

$$\frac{\partial}{\partial r}(\text{Imag } \tilde{q}_\lambda)\Big|_{e^{is}} = -\frac{\partial}{\partial s}(\text{Real } \tilde{q}_\lambda)\Big|_{e^{is}} = -q_\lambda'(s)$$

$$= \Lambda f_\lambda(s) \quad , \qquad (1.25)$$

for all $s \in (-\pi,\pi)$.

Before finishing this discussion of conformal mappings we note that, in the limiting case when $\lambda \to \infty$, a mapping which takes the region $R_\infty = \{\chi + i\eta : \chi \in (-\infty,\infty) \;\; -h < \eta < 0\}$ conformally onto $\mathcal{D}'$, and the boundary portion $\Gamma_\infty = \{\chi + i0 : \chi \in (-\infty,\infty)\}$ onto $\{e^{is} : -\pi < s < \pi\}$, is given by

$$\tilde{p}(\zeta) = -\tanh^2(\pi(\zeta + ih)/4h)$$

$$= -s_\infty^2(\pi(\zeta + ih)/4h) \quad .$$

If the inverse of $\tilde{p}$ is denoted by $\tilde{q}$, then it follows just as before that

$$\frac{\partial}{\partial r}(\text{Imag } \tilde{q})\Big|_{e^{is}} = -\frac{\partial}{\partial s}(\text{Real } \tilde{q})\Big|_{e^{is}}$$

$$= \frac{h}{\pi} \sec\frac{s}{2} = \frac{2h}{\pi} f(s) \quad . \qquad (1.26)$$

If $v : [-\pi,\pi] \to \mathbb{R}$ is continuous, odd, periodic, and $v(\pi) = 0$, then there exists a unique harmonic function u on the unit disc $\mathcal{D} = \{\xi : |\xi| < 1\}$ which satisfies the Neumann boundary condition $\partial u/\partial r|_{e^{is}} = v(s)$, $s \in (-\pi,\pi]$, and the normalization condition $\int_{-\pi}^{\pi} u(e^{is})ds = 0$. It is easy to see that for all $s \in (-\pi,\pi]$,

$$u(e^{is}) = \int_{-\pi}^{\pi} G(s,t)v(t)dt \quad ,$$

where

$$G(s,t) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin ks \sin kt}{k}$$

$$= \frac{1}{2\pi} \ell n \left| \frac{\sin((s+t)/2)}{\sin((s-t)/2)} \right| \qquad (1.27)$$

for all $(s,t) \in (-\pi,\pi] \times (-\pi,\pi]$, $s \neq t$, and u is zero on the real axis.

The next theorem concerns the change of variables which enables the convergence result of section 2.3 to be deduced from the work of [1].

THEOREM 1.1. _Let_ $V : [-\lambda/2,\lambda/2) \to \mathbb{R}$ _be a continuous, odd function which is positive on_ $(0,\lambda/2)$, _and such that_ $V(-\lambda/2) = 0$. _Then putting_ $v(s) = -V(q_\lambda(s))$, _for all_ $s \in (-\pi,\pi]$, _defines a continuous, odd function which is_ positive on $(0,\pi)$, and $v(\pi) = 0$.

Moreover, if Λ _is given by_ (1.23), _then_

$$u(s) = \Lambda \int_{-\pi}^{\pi} G(s,t)v(t)f_\lambda(t)dt \qquad (1.28)$$

for all $s \in [-\pi,\pi]$, _if and only if_

$$u(s) = -U(q_\lambda(s)) \quad ,$$

where

$$U(\chi) = \int_{-\lambda/2}^{\lambda/2} \frac{1}{2\pi} \ell n \left| \frac{s_\lambda(2K_\lambda(\chi+\varepsilon)/\lambda)}{s_\lambda(2K_\lambda(\chi-\varepsilon)/\lambda)} \right| V(\varepsilon)d\varepsilon \quad . \qquad (1.29)$$

Furthermore there exists a function $\tilde{U}$ _harmonic on_ R_λ _which is such that_

$$\tilde{U}(\chi+i0) = U(\chi), \quad \text{for } x \in [-\lambda/2,\lambda/2) \quad ,$$

$$\left.\frac{\partial \tilde{U}}{\partial \eta}\right|_{\chi+i0} = V(\chi), \quad \text{for } x \in [-\lambda/2,\lambda/2) \quad ,$$

and $\tilde{U} = 0$ _on_ $\partial R_\lambda/\{\text{Imag } \zeta = 0\}$.

Proof. It follows from (1.19) and from the formula for the elliptic function of a sum that, under the change of variables

$$\chi = q_\lambda(s) \quad \text{and} \quad \varepsilon = q_\lambda(t),\ s,t \in (-\pi,\pi] \quad ,$$

the kernel

$$\frac{1}{2\pi}\,\ln\left|\frac{s_\lambda(2K_\lambda(\chi+\varepsilon)/\lambda)}{s_\lambda(2K_\lambda(\chi-\varepsilon)/\lambda)}\right|$$

becomes

$$\frac{1}{2\pi}\,\ln\left|\frac{\sin((s+t)/2)}{\sin((s-t)/2)}\right| \quad .$$

Since $q_\lambda'(s) = -\Lambda f_\lambda(s)$ on $(-\pi,\pi]$, the result from the first part of the theorem is immediate.

Because v is continuous and odd on $(-\pi,\pi]$ and $v(\pi) = 0$, it follows that there exists a function $\tilde{u}$, harmonic on $\mathcal{D}$ and such that

$$\left.\frac{\partial\tilde{u}}{\partial r}\right|_{e^{it}} = \Lambda f_\lambda(t)v(t)$$

and

$$\tilde{u}(e^{it}) = u(t) \quad ,$$

for all $t \in (-\pi,\pi]$. Therefore if U is defined by

$$\tilde{U}(\zeta) = -\tilde{u}(\tilde{p}_\lambda(\zeta)) \quad ,$$

for all $\zeta \in R_\lambda$, then, because of (1.25), the conclusion of the theorem holds.

q.e.d.

LEMMA 1.2. <u>For all</u> $\chi,\varepsilon \in [-\lambda/2,\lambda/2] \times [-\lambda/2,\lambda/2]$, $\chi \neq \varepsilon$,

$$\frac{1}{2\pi}\,\ln\left|\frac{s_\lambda(2K_\lambda(\chi+\varepsilon)/\lambda)}{s_\lambda(2K_\lambda(\chi-\varepsilon)/\lambda)}\right| = \frac{1}{\pi}\sum_{k=1}^{\infty} k^{-1}\tanh\left(\frac{2k\pi h}{\lambda}\right)\sin\left(\frac{2\pi k\chi}{\lambda}\right)\sin\left(\frac{2\pi k\varepsilon}{\lambda}\right) \quad .$$

<u>Proof</u>. This follows by a simple calculation from the expansion for $\mathrm{sn}(u,k)$ [7; page 912(20)]:

$$\ln \mathrm{sn}(u,k) = \ln\frac{2K}{\pi} + \ln\sin\frac{\pi u}{2K} - 4\sum_{k=1}^{\infty}\frac{1}{k}\,\frac{q^k}{1+q^k}\,\sin^2\frac{k\pi u}{2K} \quad .$$

where $q = e^{-\pi K'/K}$.

q.e.d.

THEOREM 1.3. _The solutions of the linear characteristic value problem_

$$u(s) = \frac{\mu}{3}\int_{-\pi}^{\pi} G(s,t)f_{\lambda}(t)u(t)dt$$

consists precisely of the set of characteristic values $\{\frac{6\Lambda\pi k}{\lambda}\coth(\frac{2k\pi h}{\lambda})\}_{k=1}^{\infty}$ _with corresponding set of eigenfunctions_ $\left\{\sin\frac{2\pi k\sigma_{\lambda}(s)}{\lambda}\right\}_{k=1}^{\infty}$. _In particular, the smallest characteristic value is_ $\frac{6\Lambda\pi}{\lambda}\coth(\frac{2\pi h}{\lambda}) \to \frac{6}{\pi}$ _as_ $\lambda \to \infty$.

Proof. From Lemma 1.2 it follows that the set of characteristic values of the operator defined by the right-hand-side of (1.29) comprise the set $\{\frac{2\pi k}{\lambda}\coth(\frac{2\pi kh}{\lambda})\}_{k=1}^{\infty}$, and the corresponding eigenvectors are $\{\sin(2\pi kX/\lambda)\}_{k=1}^{\infty}$. The result is then an immediate consequence of Theorem 1.1, and the fact that $\Lambda \to 2h/\pi$ as $\lambda \to \infty$ (see (1.24)).

q.e.d.

1.4. On integral equations for water-waves

The purpose of this section is to show the equivalence of two nonlinear integral equations, each of which is a formulation of the periodic water-wave problem when the mean depth and the wavelength are given. Theorem 1.4 is a statement of this equivalence, while in Theorem 1.5 a precise description of the wave which corresponds to a solution of equation (1.31) is given. Theorem 1.6, which is taken without proof from [1], is a statement of the corresponding result for solitary waves.

Let h be fixed, as in the previous section, and let λ be any positive real number.

THEOREM 1.4(a). _If_ $\Theta : [-\lambda/2,\lambda/2] \to \mathbb{R}$ _is continuous, odd, and_ $0 < \Theta(X) < \pi/2$ _on_ $(0,\lambda/2)$, _and if for all_ $s \in [-\pi,\pi]$

$$\theta(s) = -\Theta(q_{\lambda}(s)) \quad , \tag{1.30}$$

then $\theta : [-\pi,\pi] \to \mathbb{R}$ is continuous, odd, and $0 < \theta(s) < \pi/2$ on $[0,\pi]$. Moreover, for some $\mu > 0$, θ satisfies the equation

$$\theta(s) = \frac{1}{6}\int_{-\pi}^{\pi} \frac{1}{\pi}\,\ell n\left|\frac{\sin((s+t)/2)}{\sin((s-t)/2)}\right| \frac{f_\lambda(t)\sin\theta(t)}{\frac{1}{\mu} + \int_0^t f_\lambda(w)\sin\theta(w)\,dw}\,dt \tag{1.31}$$

for all $s \in [-\pi,\pi]$, if and only if Θ satisfies the equation

$$\Theta(\chi) = \frac{1}{6}\int_{-\lambda/2}^{\lambda/2} \frac{1}{\pi}\,\ell n\left|\frac{s_\lambda(2K_\lambda(\chi+\varepsilon)/\lambda)}{s_\lambda(2K_\lambda(\chi-\varepsilon)/\lambda)}\right| \frac{\sin\Theta(\varepsilon)}{\Lambda/\mu + \int_0^\varepsilon \sin\Theta(w)\,dw}\,d\varepsilon \tag{1.32}$$

for all $\chi \in [-\lambda/2,\lambda/2]$. Here Λ is given by (1.23).

(b) If Θ is as in (a) and satisfies (1.32) then there exists a harmonic function on R_λ which coincides with Θ on the boundary portion Γ_λ', and which is zero on $\partial R_\lambda \backslash \Gamma_\lambda'$. If $\tilde{\Theta}$ is used to denote this harmonic function on R_λ, then

$$\left.\frac{\partial\tilde{\Theta}}{\partial\eta}\right|_{\chi+i0} = \frac{1}{3}\,\frac{\sin\Theta(\chi)}{\frac{\Lambda}{\mu} + \int_0^\chi \sin\Theta(w)\,dw} \tag{1.33}$$

for all $\chi + i0 \in \Gamma_\lambda'$.

Proof. This result is immediate from Theorem 1.1 and equation (1.25).

q.e.d.

In order to use the methods of [1] to prove that connected sets of periodic waves converge to solitary waves in the long-wave limit it is necessary to be explicit about the waves to which solutions of (1.31) correspond.

THEOREM 1.5. Suppose that θ is an odd, continuous function on $[-\pi,\pi]$ with $0 < \theta(s) \leq \pi$ on $(0,\pi)$ and $\theta(\pi) = 0$, which satisfies the integral equation (1.31) on $[-\pi,\pi]$ for some $\mu > 0$. Then θ is analytic on $(-\pi,\pi)$ and satisfies $0 < \theta(s) < \pi/2$ on $(0,\pi)$. Moreover there exists a solution of the

periodic water-wave problem of period λ on a flow of mean depth h. The mean velocity of the flow is given by

$$c = \frac{\sqrt{3g}}{\Lambda}\left(\frac{2}{\lambda}\int_0^\pi \frac{f_\lambda(t)\cos\theta(t)}{\{\frac{1}{\mu}+\int_0^t f_\lambda(w)\sin\theta(w)\,dw\}^{1/3}}\,dt\right)^{-\frac{3}{2}} , \tag{1.34}$$

from which the flux Q and the velocity at the crest q_c may be calculated as follows:

$$Q = ch \tag{1.35}$$

and

$$q_c = \left(\frac{3g\Lambda c}{\mu}\right)^{1/3} . \tag{1.36}$$

The free surface Γ_λ is then given by $\{(x,H_\lambda(x)) : x \in (-\lambda/2,\lambda/2)$, where

$$H_\lambda(x) - H_\lambda(0) = -\left(\frac{c^2\Lambda^2}{3g}\right)^{1/3} \int_0^{\alpha^{-1}(x)} \frac{f_\lambda(t)\sin\theta(t)}{(\frac{1}{\mu}+\int_0^t f_\lambda(w)\sin\theta(w)\,dw)^{1/3}}\,dt \tag{1.37}$$

and

$$\alpha(s) = -\left(\frac{c^2\Lambda^2}{3g}\right)^{1/3} \int_0^{s} \frac{f_\lambda(t)\cos\theta(t)}{(\frac{1}{\mu}+\int_0^t f_\lambda(w)\sin\theta(w)\,dw)^{1/3}}\,dt . \tag{1.38}$$

Proof. The method of proof of Theorem 2.2 (ii), (iii) applies to any solution of (1.31), and not just to those in C. Hence the analyticity of θ and the a priori bound of $\pi/2$ for θ follow immediately.

Let $\tilde\Theta$ be the function which is harmonic in R_λ mentioned in Theorem 1.4(b), and let $\tilde T$ denote the unique function which is harmonic in R_λ and conjugate to Θ (that is, $\tilde T - i\tilde\Theta$ is analytic in R_λ) such that

$$\frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} \exp(\tilde T(X - ih)\,dX = 1 . \tag{1.39}$$

By Lewy's theorem [13] $\tilde{T} - i\Theta$ is an analytic function on $\bar{R}_\lambda$, and we can use it to define an analytic function $\tilde{m}$ on $\bar{R}_\lambda$ by putting

$$\tilde{m}(\zeta) = \int_{-ih}^{\zeta} \exp(\tilde{T}(\zeta') - i\tilde{\Theta}(\zeta'))d\zeta' \quad . \tag{1.40}$$

If $\tilde{m}(\zeta_1) = \tilde{m}(\zeta_2)$ for $\zeta_1, \zeta_2 \in R_\lambda$, then

$$\int_{\zeta_1}^{\zeta_2} \exp(\tilde{T}(\zeta'))\cos \tilde{\Theta}(\zeta')d\zeta' = 0 \quad ,$$

which is false (unless $\zeta_1 = \zeta_2$) since, by the maximum principle, $|\tilde{\Theta}| < \pi/2$ in R_λ. Since $\tilde{m}'(\zeta) \neq 0$ in R_λ, it follows that $\tilde{m}$ maps R_λ conformally onto a region S_λ in the z-plane, and is invertable. The function $\tilde{T}$ is even in χ since $\tilde{\Theta}$ is odd in χ, and, because $\tilde{\Theta} = 0$ on $\partial R_\lambda \backslash \Gamma_\lambda'$, it follows that S_λ is the region bounded by the line $y = 0$, $x = \pm\lambda/2$ and the curve $\Gamma_\lambda = \tilde{m}(\Gamma_\lambda')$. It is clear that, for some function H_λ, $\Gamma_\lambda = \{x + iH_\lambda(x) : x \in (-\lambda/2,\lambda/2)\}$ and $H_\lambda'(x) = -\tan \Theta(\tilde{m}^{-1}(x + iH_\lambda(x)))$.

Since $\tilde{m}$ is invertible, a complex potential $\omega = \phi + i\psi$ can be defined on S_λ by putting

$$\phi(z) + i\psi(z) = \omega(z) = c\tilde{m}^{-1}(z) \tag{1.41}$$

where c is given by (1.34). It remains to show that for this choice of ϕ, ψ and c, all of the boundary conditions (1.1) - (1.10) are satisfied in S_λ.

The velocity field (u,v) generated in S_λ by the complex potential ω is given by

$$u(z) - iv(z) = -\frac{d\omega}{dz}$$

$$= -c \exp(-\tilde{T}(\tilde{m}^{-1}(z)))(\cos \tilde{\Theta}(\tilde{m}^{-1}(z)) + i \sin \tilde{\Theta}(\tilde{m}^{-1}(z))) \quad , \tag{1.42}$$

whence $-\tilde{\Theta}(\tilde{m}^{-1}(z))$ is the angle which the negative velocity vector makes with

the x-axis, and $c\exp(-\tilde{T}(\tilde{m}^{-1}(z)))$ is the speed of the flow at $z \in S_\lambda$. Moreover, the mean velocity along the bottom is

$$\frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} -c\exp(-\tilde{T}(\tilde{m}^{-1}(z)))dz$$

$$= \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} -c\, d\,\zeta = -c \quad . \tag{1.43}$$

Let $T : [-\lambda/2,\lambda/2] \to \mathbb{R}$ denote the restriction of $\tilde{T}$ to Γ_λ'. Then, since $\Theta = 0$ on $\partial R_\lambda \backslash \Gamma_\lambda'$, it follows, by Cauchy's Theorem and (1.39), that

$$\int_{-\lambda/2}^{\lambda/2} \exp(T(X))\cos\Theta(X)dX = \int_{-\lambda/2}^{\lambda/2} \exp(\tilde{T}(X - ih))dX = \lambda \quad . \tag{1.44}$$

However, from (1.33) and the Cauchy-Riemann equations,

$$T(X) = T(0) - \frac{1}{3}\,\ell n(1 + (\mu/\Lambda)\int_0^X \sin\Theta(w)dw) \tag{1.45}$$

for all $X \in [-\lambda/2,\lambda/2]$. Substituting this expression for T into (1.44) gives

$$\exp(-T(0)) = \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} \frac{\cos\Theta(X)}{(1 + (\mu/\Lambda)\int_0^X \sin\Theta(w)dw)^{1/3}}\,dX$$

$$= \frac{1}{\lambda}\int_{-\pi}^{\pi} \frac{\Lambda f_\lambda(t)\cos\theta(t)}{(1 + \mu\int_0^t f_\lambda(w)\sin\theta(w)dw)^{1/3}}\,dt$$

$$= \left(\frac{3g\Lambda}{\mu c^2}\right)^{1/3} \quad , \tag{1.46}$$

by (1.34), whence

$$q_c^3 = c^3\exp(-3T(0)) = \frac{3g\Lambda c}{\mu} \quad .$$

It is clear that S_λ and the flow generated by the choice of velocity potential given in (1.41) satisfies (1.1) - (1.5), and that the mean velocity of the flow is $-c$, where c is calculated from (1.34). Since $\tilde{m}^{-1}(z) = -ih$ for z on the bottom of S_λ, and $\tilde{m}^{-1}(z)$ is real on Γ'_λ, it follows that $\psi = -ch$ on the bottom and $\psi = 0$ on the top. Thus (1.7) - (1.9) are satisfied. Provided that the free surface condition given by (1.10) can be verified, it has been shown that solutions of the integral equation (1.31) correspond to solutions of the free boundary-value problem for waves of wavelength λ on a flow of mean depth h, the mean velocity being given by (1.34). Now, for all $\chi \in [-\lambda/2, \lambda/2]$,

$$\frac{d}{d\chi}\left[\frac{1}{2}c^2 \exp(-2T(\chi)) + g \operatorname{Imag} \tilde{m}(\chi + i0)\right]$$

$$= -c^2 \exp(-2T(\chi))T'(\chi) - g \exp(T(\chi))\sin\Theta(\chi)$$

$$= \exp(T(\chi))\{-c^2 \exp(-3T(\chi)T'(\chi) - g \sin\Theta(\chi)\}$$

$$= 0$$

by (1.45), (1.46). Hence, since $\exp(-T(m^{-1}(z)))$ is the speed of the flow at any point $z \in \bar{S}_\lambda$, the boundary condition (1.10) is satisfied on Γ_λ.

Finally, to calculate the wave profile we proceed as follows. At a point $x + iy \in \Gamma_\lambda$, the free surface is given by

$$y = H_\lambda(x) \quad ,$$

and $H'_\lambda(x) = -\tan\Theta(\tilde{m}^{-1}(x + iH_\lambda(x)))$. Hence

$$H_\lambda(x) - H_\lambda(0) = \int_0^x H'_\lambda(w)\,dw$$

$$= -\int_0^{\tilde{\alpha}^{-1}(x)} \tan\Theta(\chi)\cos\Theta(\chi)\exp(T(\chi))\,d\chi$$

$$= -\left(\frac{\mu c^2}{3g\Lambda}\right)^{1/3} \int_0^{\tilde{\alpha}^{-1}(x)} \frac{\sin \Theta(\chi)}{(1 + (\mu/\Lambda)\int_0^{\chi} \sin \Theta(w)\,dw)^{1/3}}\, d\chi$$

where $\tilde{\alpha} : (-\lambda/2, \lambda/2) \to \mathbb{R}$ is given by

$$\tilde{\alpha}(\chi) = \text{Real } \tilde{m}(\chi + i0)$$

$$= \left(\frac{\mu c^2}{3g\Lambda}\right)^{1/3} \int_0^{\chi} \frac{\cos \Theta(\chi')}{(1 + (\mu/\Lambda)\int_0^{\chi'} \sin \Theta(w)\,dw)^{1/3}}\, d\chi' \quad .$$

Hence

$$H_\lambda(x) - H_\lambda(0) = -\left(\frac{\Lambda^2 c^2}{3g}\right)^{1/3} \int_0^{\alpha^{-1}(x)} \frac{f_\lambda(t) \sin \theta(t)}{(\frac{1}{\mu} + \int_0^t f_\lambda(w) \sin \theta(w)\,dw)^{1/3}}\, dt$$

where

$$\alpha^{-1}(x) = (\tilde{\alpha} \circ q_\lambda)^{-1}(x) = p_\lambda(\tilde{\alpha}^{-1}(x))$$

and so

$$\alpha(s) = \tilde{\alpha} \circ q_\lambda(s) = \left(\frac{\mu c^2}{3g\Lambda}\right)^{1/3} \int_0^{q_\lambda(s)} \frac{\cos \Theta(\chi')}{(1 + (\frac{\mu}{\Lambda})\int_0^{\chi'} \sin \Theta(w)\,dw)^{1/3}}\, d\chi'$$

$$= -\left(\frac{\Lambda^2 c^2}{3g}\right)^{1/3} \int_0^{s} \frac{f_\lambda(t) \cos \theta(t)}{(\frac{1}{\mu} + \int_0^t f_\lambda(w) \sin \theta(w)\,dw)^{1/3}}\, dt \quad .$$

This completes the proof of the theorem.

q.e.d.

THEOREM 1.6. <u>Suppose that</u> θ <u>is an odd, continuous function on</u> $[-\pi, \pi]$ <u>with</u> $0 < \theta(s) < \pi/2$ <u>on</u> $(0, \pi)$ <u>and</u> $\theta(\pi) = 0$, <u>which satisfies the integral equation</u>

$$\theta(s) = \frac{1}{6} \int_{-\pi}^{\pi} \frac{1}{\pi} \ln \left| \frac{\sin((s+t)/2)}{\sin((s-t)/2)} \right| \frac{f(t) \sin \theta(t)}{\frac{1}{\mu} + \int_0^t f(w) \sin \theta(w)\,dw}\, dt \quad , \qquad (1.47)$$

for all $s \in [-\pi,\pi]$ *where* $\mu > 0$ *and* $f(t) = \frac{1}{2}\sec\frac{t}{2}$ *for* $t \in (-\pi,\pi)$.
Then $\theta f \in L_1(-\pi,\pi)$, θ *is analytic on* $(-\pi,\pi)$ *and* $0 < \theta(s) < \pi/2$ *on* $(0,\pi)$.
Moreover, if h *and* c *are any positive real numbers which satisfy*

$$\frac{6gh}{\pi c^2}\left(\frac{1}{\mu} + \int_0^{\pi} f(w)\sin\theta(w)\,dw\right) = 1 ,$$

then there exists a steady solitary wave flow whose mean velocity and asymptotic height (see section 1.2) *are* $-c$ *and* h *respectively. The velocity* q_c *of the flow at the wave crest may be calculated from the expression*

$$\frac{\pi q_c^3}{6ghc} = \frac{1}{\mu} .$$

Moreover the solitary wave profile Γ *is given by* $\{(x,H(x)) : x \in \mathbb{R}\}$ *where*

$$H(x) - H(0) = -\frac{h}{3}\left(\frac{36c^2}{\pi^2 gh}\right)^{1/3}\int_0^{\alpha^{-1}(x)} \frac{f(t)\sin\theta(t)}{\left(\frac{1}{\mu} + \int_0^t f(w)\sin\theta(w)\,dw\right)^{1/3}}\,dt \tag{1.48}$$

and

$$\alpha(s) = -\frac{h}{3}\left(\frac{36c^2}{\pi^2 gh}\right)^{1/3}\int_0^{s} \frac{f(t)\cos\theta(t)}{\left(\frac{1}{\mu} + \int_0^t f(w)\sin\theta(w)\,dw\right)^{1/3}}\,dt . \tag{1.49}$$

Proof. While this theorem is formally the limiting case of Theorem 1.5 as $\lambda \to \infty$, it needs a separate proof. This may be done by modifying the method of proof of Theorem 1.5, using the mapping p from R_∞ onto $\mathcal{D}'$ introduced in section 1.3. The function Θ is then required to be in $L_1(-\infty,\infty)$, odd, positive on $(0,\infty)$, and to satisfy

$$\Theta(\chi) = \frac{1}{6}\int_{-\infty}^{\infty}\frac{1}{\pi}\,\ell n\left|\frac{\tanh(\pi(\chi+\varepsilon)/4h)}{\tanh(\pi(\chi-\varepsilon)/4h)}\right|\frac{\sin\Theta(\varepsilon)}{\frac{2h}{\pi\mu} + \int_0^{\varepsilon}\sin\Theta(w)\,dw}\,d\varepsilon ,$$

an equation which may be obtained from equation (1.47) by putting

$\lambda = \ell n(\sec \frac{s}{2} + \tan \frac{s}{2})$, and $\varepsilon = \ell n(\sec \frac{t}{2} + \tan \frac{t}{2})$, $s,t \in (-\pi,\pi)$. An alternative proof is to be found in [1; Theorems 1.1, 4.1 and 4.3]. (The function Θ in [1; Theorem 1.1] differs from that which arises in the method suggested by the proof of Theorem 1.5 by a change of sign.)

For the sake of giving a complete description of Nekrasov's integral equations we include, in the Appendix, the equation for periodic waves on a flow which is infinitely deep. The derivation there is slightly different from those already in the literature, and emphasizes the dependence of the flow parameters on a given solution of the equation. It is shown how this equation can be written in an alternative form which involves the conjugate operator from the L_2-theory of Fourier series. While a similar formulation might be adopted for the periodic problem on a flow of <u>finite</u> depth (see [10]), we avoid this approach because the normalization requirement ([10]; p. 1002, (1.19)) means that when the depth is finite the conjugate operator is nonlinear. In any case (1.31) and (1.32) are preferred, since the dependence of the integrand on θ and Θ is given explicitly.

2. THE GLOBAL THEORY

2.1. Background

The first proof of the existence of large amplitude, periodic water-waves is due to Krasovskii [10], and is based on an application of the monotone minorant theorem [9] to a particular version of Nekrasov's integral equation. Among his results on the existence of periodic water-waves in a channel with a wave-like bottom is included the special case when the bottom is flat. In this case the conclusion is that, for each positive h and λ, and for each $\beta \in (0,\pi/6)$, there exists a wave of wavelength λ, on a flow whose mean depth is h, which is such that the maximum angle of inclination of the free surface to the horizontal is β and the mean velocity of all such waves is bounded away from zero and infinity. Though this result is highly suggestive it does not amount to a global bifurcation theorem since neither the question of bifurcation, nor the question of the existence of a connected set of solutions is considered. The first result of this kind is due to Keady and Norbury [8], who regard Nekrassov's integral equation as an example in the general theory of global bifurcation [6], [7], [22]. Their result is the following: if L and Q are fixed, positive real numbers, then there exists a connected set of periodic water-waves which bifurcates from the set of horizontal, uniform flows, and each of which is of flux Q, and each of which has wavelength $2L$ with respect to the velocity potential. This set contains a wave whose velocity at the crest is q_c for any value of q_c in the interval $(0,(\frac{gL}{\pi}\tanh(\frac{\pi Q}{L}))^{1/3})$.

Since the mathematical theory of steady water-waves still lacks any global uniqueness result, it is not possible to assert that the solutions obtained by Krasovskii's method are included in the connected set which Keady and Norbury obtain. (In principle, Krasovskii's method may yield solutions lying off the bifurcating set, is such exist.) Nevertheless it can be shown [21] (independently

of the work of Krasovskii), that this bifurcating set contains waves with maximum angle of inclination to the horizontal β, for all values of β in the interval $(0,\pi/6)$. Indeed, it has been shown by McLeod [14] that this connected set of water waves contains a wave whose maximum angle of inclination to the horizontal is β, for all $\beta \in (0,\pi/6+\varepsilon)$ for some $\varepsilon > 0$.

In the next section we shall summarize the global bifurcation theory for periodic water-waves <u>of spatial wavelength</u> λ <u>on a flow of mean depth</u> h. Because of our declared intention to deduce from these results the corresponding theorems for solitary waves on a flow of mean depth h, we state theorems about the periodic problem in terms of the integral equation (1.31) rather than the equivalent equation (1.32).

2.2. <u>The bifurcation of periodic waves of wavelength</u> λ <u>on a flow of mean depth</u> h

Throughout this section we consider waves of wavelength λ on a flow of <u>fixed</u> mean depth h. Accordingly, we are interested in solutions (μ,θ) of (1.31) with $\mu > 0$ and $0 < \theta(s) < \pi/2$ on $(0,\pi)$. Since all solutions of (1.31) are odd, it suffices instead to consider the eigenvalue problem

$$\theta(s) = \frac{2}{3}\int_0^\pi G(s,t)\frac{f_\lambda(t)\sin\theta(t)}{\frac{1}{\mu}+\int_0^t f_\lambda(w)\sin\theta(w)\,dw}\,dt \tag{2.1}$$

where the kernel G is defined in (1.27). Let $C_0[0,\pi]$ denote the Banach space of continuous functions on $[0,\pi]$ which vanish at 0 and π, and let K_0 denote the closed, reproducing cone of non-negative functions in $C_0[0,\pi]$. For any $[a,b] \subset [0,\pi]$, $C[a,b]$ denotes the usual Banach space of continuous functions on $[a,b]$, with the supremum norm. Since G is non-negative almost everywhere on $[0,\pi]\times[0,\pi]$ and is the kernel of a compact, linear Hammerstein

operator on $C_0[0,\pi]$, it follows that this linear operator leaves K_0 invariant. The linearization of (2.1) about $\theta = 0$ is given by

$$\theta(s) = \frac{2\mu}{3}\int_0^{\pi} G(s,t)f_{\lambda}(t)\theta(t)dt \quad , \tag{2.2}$$

and from Lemma 1.3 it follows that the characteristic value with smallest absolute value is $6\Lambda\pi\lambda^{-1}\coth(2\pi h/\lambda) \downarrow 6/\pi$ as $\lambda \to \infty$, the corresponding eigenvector being $\sin(2\pi q_{\lambda}(s)/\lambda)$. Before the global bifurcation may be stated, on further observation is necessary.

LEMMA 2.1. *Let* $\mu > 0$, *and let* $\theta \in K_0$ *be such that, for all* $s \in [0,\pi]$,

$$\theta(s) = \frac{2}{3}\int_0^{\pi} G(s,t)\,\frac{f_{\lambda}(t)\sin(J\theta(t))}{\frac{1}{\mu} + \int_0^t f_{\lambda}(w)\sin(J\theta(w))dw}\,dt \quad , \tag{2.3}$$

where

$$Jx = (\operatorname{sgn} x)\min\{|x|,\pi\}, \quad \text{for all } x \in \mathbb{R} \quad .$$

Then (i) $0 < \theta(s) < \pi$ *on* $(0,\pi)$, *and*

(ii) $\mu > 6\Lambda\pi\lambda^{-1}\coth(2\pi h/\lambda)$.

Proof. The proof of this result is an easy consequence of the maximum principle, and is proved by the method used to establish [1; Theorem 3.3(a), (c)]. No modifications are required.

q.e.d.

The next result is a summary of the global existence theory for solutions of equation (1.31). Throughout the discussion, the mean depth is fixed. Let $S_{\lambda} = \{(\mu,\theta) \in (0,\infty) \times K_0 : (\mu,\theta)$ satisfies (1.31) and $\theta \neq 0\} \cup \{(\mu_{\lambda},0)\}$ where $\mu_{\lambda} = 6\Lambda\pi\lambda^{-1}\coth(2\pi h/\lambda)$.

THEOREM 2.2. ([8], [10], [13], [14], [21]) *Let* C_{λ} *denote the maximal connected subset of* S_{λ} *in* $\mathbb{R} \times C_0[0,\pi]$ *which contains* $(\mu_{\lambda},0)$. *Then*

(i) $\mathcal{C}_\lambda$ is closed and unbounded;

(ii) if $(\mu,\theta) \in \mathcal{C}_\lambda \setminus \{(\mu_\lambda,0)\}$, then $\mu > \mu_\lambda$ and $0 < \theta(s) < \pi/2$ on $(0,\pi)$, whence $\{\mu : (\mu,\theta) \in \mathcal{C}_\lambda\} = [\mu_\lambda,\infty)$.

(iii) θ is a real analytic function on $[0,\pi]$.

(iv) For each λ, $\delta > 0$, there exists a constant $\beta_{\lambda,\delta}$ such that

$$\theta(s) \geq \beta_{\lambda,\delta} \sin s \tag{2.4}$$

if $\mu > \mu_\lambda + \delta$, and $(\mu,\theta) \in \mathcal{C}_\lambda$.

(v) If $(\mu,\theta) \in \mathcal{C}_\lambda$, then the mean velocity of the wave, $c(\mu,\theta)$, is given by the formula (1.34). For each $\lambda > 0$, there exists a closed interval $[a_\lambda,b_\lambda] \subset (0,\pi)$ such that

$$\{c(\mu,\theta) : (\mu,\theta) \in \mathcal{C}_\lambda\} \subset [a_\lambda,b_\lambda] \quad ,$$

and

$$\bigcup_{\lambda>0} [a_\lambda,b_\lambda] = (0,M]$$

for some $M > 0$.

Let the velocity of the corresponding flow at the wave crest, calculated from (1.36), be denoted by $q_c(\mu,\theta)$, $(\mu,\theta) \in \mathcal{C}_\lambda$.

(vi) If $\{(\mu_n,\theta_n)\} \subset \mathcal{C}_\lambda$ and $\mu_n \to \infty$ as $n \to \infty$, then $q_c(\mu_n,\theta_n) \to 0$, and there exists a subsequence $\{\theta_{n(k)}\}$ of $\{\theta_n\}$ such that $\theta_{n(k)} \to \theta$ uniformly on $[\delta,\pi]$ for each $\delta > 0$, where θ is a non-trivial solution of the equation

$$\theta(s) = \frac{2}{3}\int_0^\pi G(s,t) \frac{f_\lambda(t)\sin\theta(t)}{\int_0^t f_\lambda(w)\sin\theta(w)\,dw}\,dt \quad . \tag{2.5}$$

Furthermore $\lim_{s\to 0+} \theta(s) = a > 0$ and the following dichotomy holds: either $\lim_{s\to 0+} \theta(s) = \pi/6$, or

$$\varliminf_{s\to 0+} \theta(s) < \pi/6 < \varlimsup_{s\to 0+} \theta(s) \quad .$$

The periodic wave corresponding to a solution of (2.5) has a stagnation point at its crest (i.e. $q_c = 0$).

(vii) Let $\{(\mu_n,\theta_n)\} \subset C_\lambda$ be such that $\mu_n \to \infty$. Since (μ_n,θ_n) satisfies equation (1.31), for each n, it follows that the function θ_n^* defined on $[0,\mu_n\pi]$

$$\theta_n^*(x) = \theta_n(x/\mu_n)$$

satisfies the equation

$$\theta_n^*(x) = \frac{2}{3}\int_0^{\mu_n\pi} G(x/\mu_n, y/\mu_n)\, \frac{f_\lambda(y/\mu_n)\sin\theta_n^*(y)}{1 + \int_0^x f_\lambda(w/\mu_n)\sin\theta_n^*(w)\,dw}\, dy$$

for all $x \in [0,\mu_n\pi]$.

Moreover, as $n \to \infty$. $\{\theta_n^*\}$ converges uniformly on compact subsets of $(0,\infty)$ to a function θ^* which satisfies the boundary-layer equation

$$\theta^*(x) = \frac{2}{3}\int_0^\infty \frac{1}{2\pi}\,\ell n\left|\frac{x+y}{x-y}\right|\, \frac{\frac{1}{2}\sin\theta^*(y)}{1 + \frac{1}{2}\int_0^y \sin\theta^*(w)\,dw}\, dy \quad .$$

Since $\sup_{x\in(0,\infty)} \theta^*(x) > \pi/6$, it follows that there exists an $\varepsilon > 0$ such that, for all n sufficiently large, $|\theta_n|_{C_0[0,\pi]} \geq \pi/6 + \varepsilon$. Hence for each $\beta \in [0,\pi/6 + \varepsilon]$ there exists a periodic water wave of any specified mean depth and wavelength, the free surface of which subtends a maximum angle to the horizontal of β.

Proof. (i) The proof of this is a simple application of [6; Theorem 2] to equation (2.3), once the a priori bound of Lemma 2.1 has been noted (see [8; Lemma 4.1] for a similar treatment of equation (3.2)).

(ii) That $\mu > \mu_\lambda$ follows after multiplying equation (2.1) by the eigenfunction of the linear equation (2.2), which corresponds to the characteristic value μ_λ, and integrating over $(0,\pi)$.

A slight modification of [1; Theorem 3.3(d)] yields that $\theta(s) < \pi/2$ on $(0,\pi)$. In this case the crucial observation is that the function P defined on $\mathcal{D}'$ by putting

$$P(\zeta) = -\frac{1}{2}\exp(-2\tilde{\rho}(\zeta)) - Y(\zeta)$$

is a super-harmonic function on $\mathcal{D}'$ which attains its minimum at every point of the boundary portion $\{e^{it} : t \in (-\pi,\pi)$. Here $\tilde{\rho}$ and Y are defined as follows. If $(\mu,\theta) \in \mathcal{C}_\lambda$, then suppose that

$$\theta \sim \sum_{k=1}^{\infty} a_k \sin ks \quad ,$$

and put

$$\rho(t) = -\frac{1}{3}\ln\left(\frac{1}{\mu} + \int_0^t f_\lambda(w)\sin\theta(w)dw\right) \quad ,$$

for $t \in [-\pi,\pi]$. If $a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi}\rho(t)dt$, it follows that

$$\tilde{\omega}(re^{it}) = a_o + \sum_{k=1}^{\infty} a_k r^k e^{ikt}$$

$r \in [0,1)$, $t \in (-\pi,\pi)$ defines an analytic function on $\mathcal{D}'$. Then put

$$\tilde{\rho}(\zeta) = \text{Real } \tilde{\omega}(\zeta) \quad ,$$

and

$$Y(\zeta) = \text{Imag } \frac{1}{3\Lambda}\int_0^{\zeta} \exp(\tilde{\omega}(\hat{\zeta}))\tilde{q}_\lambda'(\hat{\zeta})d\hat{\zeta}$$

for $\zeta \in \mathcal{D}'$ where $\tilde{q}_\lambda$ is the inverse of the conformal mapping $\tilde{p}_\lambda$ introduced in section 1.2 (and prime denotes differentiation).

(iii) That θ is analytic on $[0,\pi]$ is a consequence of Lewy's theorem [13].

(iv) If this result is false, then for some $\delta > 0$ and for each n there exists $(\mu_n,\theta_n) \in \mathcal{C}_\lambda \cap [\mu_\lambda + \delta,\infty) \times K_0$, and $s_n \in (0,\pi)$ such that $\theta_n(s) \le n^{-1} \sin s_n$. Now for each closed interval $[a,b] \subset (0,\pi)$, there exists $\delta([a,b]) > 0$ such that if $t \in [a,b]$,

$$G(s,t) \geq \delta \sin s$$

for all $s \in [0,\pi]$ (see [1; Theorem 2.5(f)]). Hence for each $[a,b] \subset (0,\pi)$

$$n^{-1} \sin s_n \geq \theta_n(s_n) = \frac{2}{3}\int_0^\pi G(s_n,t)\,\frac{f_\lambda(t)\sin\theta_n(t)}{\frac{1}{\mu_n} + \int_0^t f_\lambda(w)\sin\theta_n(w)\,dw}\,dt$$

$$\geq \{\frac{2\delta}{3}\int_a^b \frac{f_\lambda(t)\sin\theta_n(t)}{\frac{1}{\mu_n} + \int_0^t f_\lambda(w)\sin\theta_n(w)\,dw}\,dt \sin s_n \quad .$$

Since $[a,b]$ is chosen arbitrarily in $(0,\pi)$, there results that

$$\frac{f_\lambda(t)\sin\theta_n(t)}{\frac{1}{\mu_n} + \int_0^t f_\lambda(w)\sin\theta_n(w)\,dw} \to 0$$

almost everywhere in $[0,\pi]$. From the a priori bound established in (ii), it follows that

$$\theta_n(t) \to 0$$

almost everywhere in $[0,\pi]$. However, an integration of (2.1) over $(0,\pi)$ after multiplication by $\sin s$ yields that

$$\int_0^\pi \theta_n(s)\sin s\,ds = \frac{1}{3}\int_0^\pi \frac{f_\lambda(t)\sin\theta_n(t)\sin t}{\frac{1}{\mu_n} + \int_0^t f_\lambda(w)\sin\theta_n(w)\,dw}\,dt$$

$$\geq \frac{1}{3\pi}\,\frac{\int_0^\pi \theta_n(t)\sin t\,dt}{\frac{1}{\mu_n} + \int_0^\pi f_\lambda(w)\sin\theta_n(w)\,dw} ,$$

whence $\{\mu_n\}$ is bounded, since $\theta_n \to 0$ almost everywhere. Because G is the kernel of a compact linear operator on $C_0[0,\pi]$, and because $\{\mu_n\}$ is bounded, it follows that θ_n converges to 0 in $C_0[0,\pi]$. But C_λ is closed, from which there follows the contradiction that the sequence $\{\mu_n\} \subset [\mu_\lambda + \delta, \infty)$ converges to μ_λ.

(v) If $(\mu,\theta) \in C_\lambda$, $\lambda > 0$, then by (1.23) and (1.34) there results that

$$c(\mu,\theta) \leq \text{const. } K_\lambda \lambda^{\frac{1}{2}} \left\{ \int_0^\pi \frac{f_\lambda(t)\cos\theta(t)}{\left(\frac{1}{\mu} + \int_0^t f_\lambda(w)\sin\theta(w)\,dw\right)^{1/3}}\,dt \right\}^{-3/2} .$$

Hence, for any $M > 0$, the set $\{c(\mu,\theta) : (\mu,\theta) \in C_\lambda,\ \lambda \in (0,M]\}$ is bounded above, or else there exists a sequence $(\mu_n,\theta_n) \in C_{\lambda_n}$, $\lambda_n \in (0,M]$ such that

$$\int_0^\pi \frac{f_{\lambda_n}(t)\cos\theta_n(t)}{\left(\frac{1}{\mu_n} + \int_0^t f_{\lambda_n}(w)\sin\theta_n(w)\,dw\right)^{1/3}}\,dt \to 0$$

as $n \to \infty$. In the latter case it follows that $\theta_n \to \pi/2$ almost everywhere on $[0,\pi]$, and this contradicts the fact that (μ_n,θ_n) satisfies (2.1) for each n and for some $\lambda_n \in (0,M]$. Hence the set $\{c(\mu,\theta) : (\mu,\theta) \in C_\lambda,\ \lambda \in (0,M]\}$ is bounded above. In order to show that an upper bound may be found which is independent of M we proceed as before by seeking a contradiction. If the result is false, then $c(\mu_n,\theta_n) \to \infty$ for some sequence $\{(\mu_n,\theta_n)\}$, where

$(\mu_n, \theta_n) \in C_{\lambda_n}$ and $\lambda_n \to \infty$. However, a slight modification of the proof of Theorem 2.3 (iv) yields that there must therefore exist a subsequence $\{(\mu_{n(k)}, \theta_{n(k)})\}$ such that $(1/\mu_{n(k)}, \theta_{n(k)}) \to (\alpha, \theta) \in [0,\infty) \times L_2(0,\pi)$, and $c(\mu_{n(k)}, \theta_{n(k)}) \to \{\frac{6gh}{\pi}(\alpha + \int_0^\pi f(t)\sin\theta(t)dt)\}^{1/2} \in [\sqrt{gh}, M']$, where M' is an absolute constant. This is a contradiction.

Finally, to show that, for fixed λ, the set $\{c(\mu,\theta) : (\mu,\theta) \in C_\lambda\}$ is bounded below by a positive constant it suffices to observe that

$$c(\mu,\theta) \geq \text{const.}\left(\int_0^\pi \left(\frac{1}{\mu} + \int_0^t f_\lambda(w)\sin\theta(w)\right)^{-1/3} dt\right)^{-3/2}$$

$$\geq \text{const.} \quad \text{(by (iv))}$$

where both constants are independent of $(\mu,\theta) \in C_\lambda$. To complete the proof, we observe that $c(\mu_\lambda, 0) = \{\frac{g\lambda}{2\pi}\tanh(\frac{2\pi h}{\lambda})\}^{1/2} \to 0$ as $\lambda \to 0$.

(vi) Since $c(\mu_n, \theta_n) \subset [a_\lambda, b_\lambda]$, and since $a_\lambda > 0$ it follows from (1.36) that $q_c(\mu_n, \theta_n) \to 0$ as $n \to \infty$. The asymptotic behaviour of $\{\theta_n\}$ as $n \to \infty$ is established by a slight modification of the arguments in [1; section 5], using (iv) to obtain the appropriate estimates. The behaviour of the limiting function θ may be analyzed by precisely the method used to establish [1; Theorem 5.2(d) - (g)].

(vii) This is the main result of [14] reformulated in terms of the equation (3.1). The proof for equation (3.1) is identical (with certain obvious modifications), and there is no need to repeat it here. (Since C_λ is a connected set in $\mathbb{R} \times C_0[0,\pi]$ which contains $(\mu_\lambda, 0)$ and a point (μ,θ) with $\sup_{s\in[0,\pi]} \theta(s) \geq \frac{\pi}{6} + \varepsilon$, it is immediate that for each $\beta \in [0, \frac{\pi}{6} + \varepsilon]$ there exists an element $(\mu,\theta) \in C_\lambda$ with $\sup_{s\in[0,\pi]} \theta(s) = \beta$.)

This completes the proof of the theorem.

q.e.d.

2.3. On the convergence of periodic waves to solitary waves in the long-wave limit

Throughout this section the mean depth h is fixed. The purpose here is to show the sense in which the sets C_λ of periodic water-waves converge to a set C' of solitary waves as the wavelength increases indefinitely. Recall from section 1.2, that each set C_λ contains exactly one point corresponding to a uniform horizontal flow of depth h, and that this point $(\mu_\lambda, 0)$ is the point at which periodic waves of wavelength λ and mean depth h bifurcate. In other words, on a flow of depth h, periodic waves of wavelength λ bifurcate from the horizontal flow when the mean velocity of the flow is $\{(g\lambda/2\pi)\tanh(2\pi h/\lambda)\}^{1/2}$. Moreover the value of μ_λ converges to $6/\pi$, as $\lambda \to \infty$.

Let U be any bounded, open set in $\mathbb{R} \times C_0[0,\pi]$ such that $(6/\pi, 0) \in U$. Then, for all λ sufficiently large, $C_\lambda \cap \partial U \neq \phi$. The next result is the main result of this paper.

THEOREM 2.3. Suppose $\{\lambda_n\} \subset \mathbb{R}$, and $\lambda_n \uparrow \infty$ as $n \to \infty$, and suppose that $C_{\lambda_n} \cap \partial U \neq \phi$ for each n. If $\{(\mu_n, \theta_n)\} \subset (0,\infty) \times K_0$ is a sequence such that $(\mu_n, \theta_n) \in C_{\lambda_n} \cap \partial U$ for each n, then the sequence $\{(\mu_n, \theta_n)\}$ is relatively compact in $[6/\pi, \infty) \times K_0$. If $\{(\mu_{n(k)}, \theta_{n(k)})\}$ is a subsequence of $\{(\mu_n, \theta_n)\}$ such that

$$(\mu_{n(k)}, \theta_{n(k)}) \to (\mu, \theta) \in [6/\pi, \infty) \times K_0 \quad , \tag{2.6}$$

then (i) $\mu > 6/\pi$, $\theta \neq 0$, and $(\mu, \theta) \in \partial U$;

(ii) (μ, θ) is a solution of the equation for solitary waves (1.47).

(iii) The sequence $\{f_{\lambda_{n(k)}} \theta_{n(k)}\}$ converges in $L_1(0,\pi)$ to $f\theta$, as $k \to \infty$.

(iv) If $c(\mu_{n(k)}, \theta_{n(k)})$ is calculated using $\lambda_{n(k)}$ instead of λ in expression (1.34), then

$$c(\mu_{n(k)},\theta_{n(k)}) = \{\frac{6\alpha h}{\pi}(\frac{1}{\mu} + \int_0^\pi f(t)\sin\theta(t)dt)\}^{1/2}$$

$$\in (1,M']$$

<u>for some absolute constant</u> M'.

(v) <u>For each</u> k, <u>the free surface may be denoted by</u> $\{(x,H_k(x)) : x \in (-\lambda_{n(k)}/2,\lambda_{n(k)}/2)$ <u>where</u> H_k <u>depends on</u> $\lambda_{n(k)},\mu_{n(k)}$ <u>and</u> $\theta_{n(k)}$ <u>according to the formulae</u> (1.37), (1.38). <u>As</u> $k \to \infty$,

$$H_k(x) - H_k(0) \to H(x) - H(0) \quad ,$$

<u>uniformly on compact intervals, where</u> $\{(x,H(x)) : x \in \mathbb{R}\}$ <u>is the profile of the solitary wave corresponding to the solution</u> (μ,θ) <u>of</u> (1.47). <u>The function</u> H <u>may be calculated from</u> (μ,θ) <u>by the formulae</u> (1.48), (1.49).

A proof of this theorem may be obtained by modifying the arguments of [1; Theorem 3.8]. The following lemmas facilitate this procedure.

LEMMA 2.4. <u>For any non-negative, bounded function</u> u <u>on</u> $[0,\pi]$, <u>whose support has full measure, and for any</u> $\alpha \geq 0$,

$$\frac{f_\lambda(t)u(t)}{\alpha + \int_0^t f_\lambda(w)u(w)dw} \geq \frac{f_\nu(t)u(t)}{\alpha + \int_0^t f_\nu(w)u(w)dw} \quad ,$$

if $\lambda \geq \nu > 0$, <u>and</u> f_λ, f_ν <u>are defined by the expression</u> (1.22).

<u>Proof</u>. Since $f_\lambda(t) \leq f_\nu(t)$ for all $t \in [0,\pi]$, when $\lambda \geq \nu$, it will suffice to show that

$$f_\lambda(t) \int_0^t f_\nu(w)u(w)dw \geq f_\nu(t) \int_0^t f_\lambda(w)u(w)dw$$

for all $t \in [0,\pi]$. In other words it will suffice to show that

$$0 \leq \int_0^t (f_\lambda(t)f_\nu(w) - f_\nu(t)f_\lambda(w))u(w)dw$$

$$= \int_0^t f_\nu(t)f_\nu(w)\left(\frac{f_\lambda(t)}{f_\nu(t)} - \frac{f_\lambda(w)}{f_\nu(w)}\right)u(w)dw \quad .$$

However, a simple calculation yields that f_λ/f_ν is increasing on $(0,\pi)$, and the proof is complete.

q.e.d.

LEMMA 2.5. <u>For each</u> $\lambda > 0$, <u>let</u> g_λ <u>denote the function defined on</u> $[0,\pi]$ <u>by putting</u>

$$g_\lambda(s) = \begin{cases} f_\lambda(s) & , \quad s \in [0,\pi-1/\lambda] \\ 0 & , \quad \text{otherwise} \quad . \end{cases} \tag{2.7}$$

<u>Then there exists a unique solution</u> $(\gamma_\lambda,\psi_\lambda)$ <u>of</u>

$$\psi(s) = \frac{2\gamma}{3}\int_0^\pi G(s,t)g_\lambda(t)\psi(t)dt \quad ,$$

<u>with</u> $(\gamma,\psi) \in [0,\infty) \times K_0$, <u>and</u> $|\psi|_{C_0[0,\pi]} = 1$. <u>Moreover</u> $\gamma_\lambda \downarrow 6/\pi$ as $\lambda \to \infty$.

<u>Proof</u>. The proof is similar to that of [1; Theorem 3.2]. Existence and uniqueness follow immediately from the general theory of u_0-positive linear operators, and that $\gamma_\lambda \downarrow 6/\pi$ follows by exactly the same argument as is used to show that $\gamma_n \downarrow 6/\pi$ in [1; Theorem 3.2].

q.e.d.

<u>Proof of Theorem 2.3</u>. Because of the obvious similarity between the problem here, and that of proving [1; Theorem 3.9], we shall limit ourselves to giving an outline of the proof. The letters (A'), (B'), (C') etc. when used below refer to those points of the proof of [1; Theorem 3.9] so labelled.

Since $\{(\mu_n,\theta_n)\} \subset \partial U \subset \mathbb{R} \times C_0[0,\pi]$ is a bounded sequence, there exists a subsequence $\{(\mu_{n(k)},\theta_{n(k)})\}$ and a corresponding sequence $\{\lambda_{n(k)}\} \subset \mathbb{R}$, such that

$$\mu_{n(k)} \to \mu \quad \text{in } \mathbb{R} , \tag{2.8)(a}$$

$$\theta_{n(k)} \rightharpoonup \theta \quad \text{weakly in } L_2(0,\pi) , \tag{2.8)(b}$$

$$\sin \theta_{n(k)} \rightharpoonup \sigma \quad \text{weakly in } L_2(0,\pi) , \tag{2.8)(c}$$

and

$$\lambda_{n(k)} \to \infty \quad \text{in } \mathbb{R} \tag{2.8)(d}$$

as $k \to \infty$. We shall show that the conclusions (i) - (v) of the theorem hold for this subsequence. For the sake of having a convenient notation, we shall henceforth use $\{\mu_n\}$, $\{\theta_n\}$, $\{\lambda_n\}$ to denote the subsequence for which (2.8) holds.

(i), (ii), (iii) An obvious adaptation of (A') - (D') yields that $\theta_n \to \theta$ and $\sin \theta_n \to \sin \theta$ in $L_2(0,\pi)$, as $n \to \infty$; that $\theta_n \to \theta$ in $C[0,\delta]$ for each $\delta \in (0,\pi)$; and that $(\mu,\theta) \in [\pi/6,\infty) \times K_0$ is a solution of (1.47). The next step is to prove that if $\theta = 0$, then $\mu = \frac{6}{\pi}$.

Now for each n,

$$\theta_n(s) = \frac{2}{3}\int_0^\pi G(s,t)\, \frac{f_{\lambda_n}(t)\sin \theta_n(t)}{\frac{1}{\mu_n} + \int_0^t f_{\lambda_n}(w)\sin \theta_n(w)\,dw}\, dt$$

$$\geq \frac{2}{3}\int_0^\pi G(s,t)\, \frac{f_{\lambda_\ell}(t)\sin \theta_n(t)}{\frac{1}{\mu_n} + \int_0^t f_{\lambda_\ell}(w)\sin \theta_n(w)\,dw}\, dt ,$$

for all $n \geq \ell$, by Lemma 2.4 and the fact that $\lambda_n \uparrow \infty$,

$$\geq \frac{2}{3} \int_0^\pi G(s,t) \frac{g_{\lambda_\ell}(t) \sin \theta_n(t)}{\frac{1}{\mu_n} + \int_0^t f_{\lambda_\ell}(w) \sin \theta_n(w)\,dw}\,dt \quad ,$$

where g_{λ_ℓ} is defined by (2.7). Therefore

$$\theta_n(s) \geq \frac{2}{3} A_{n,\ell} \left\{ \frac{\int_0^\pi G(s,t) g_{\lambda_\ell}(t) \theta_n(t)\,dt}{\frac{1}{\mu_n} + \int_0^\pi f_{\lambda_\ell}(w) \sin \theta_n(w)\,dw} \right\} , \tag{2.9}$$

for all $s \in [0,\pi]$ where

$$A_{n,\ell} = \inf_{s \in [0,\pi - 1/\lambda_\ell]} \frac{\sin \theta_n(s)}{\theta_n(s)} .$$

Now multiplying this inequality by $g_{\lambda_\ell} \psi_{\lambda_\ell}$, whose existence is guaranteed by Lemma 2.5, and integrating gives

$$\gamma_{\lambda_\ell} \int_0^\pi \theta_n(s) \psi_{\lambda_\ell}(s) g_{\lambda_\ell}(s)\,ds \geq A_{n,\ell} \left\{ \frac{\int_0^\pi g_{\lambda_\ell}(t) \psi_{\lambda_\ell}(t) \theta_n(t)\,dt}{\frac{1}{\mu_n} + \int_0^\pi f_{\lambda_\ell}(w) \sin \theta_n(w)\,dw} \right\} .$$

Thus

$$\frac{1}{\mu_n} + \int_0^\pi f_{\lambda_\ell}(w) \sin \theta_n(w)\,dw \geq A_{n,\ell}/\gamma_{\lambda_\ell} \quad ,$$

for all $n \geq \ell$. If $\theta_n \to 0$ in $L_2(0,\pi)$ as $n \to \infty$, then since $A_{n,\ell} \to 1$ as $n \to \infty$ for each fixed ℓ, there results that

$$1/\mu = \lim_{n\to\infty} 1/\mu_n \geq \gamma_{\lambda_\ell}^{-1} \quad ,$$

for each ℓ. Since $\gamma_{\lambda_\ell} \downarrow 6/\pi$, as $\ell \uparrow \infty$, it follows that $\mu \leq 6/\pi$ But (μ,θ) is a solution of (1.47) and so, by [1; Theorem 3.7], $\mu \geq 6/\pi$ We have shown that if $(\mu_n,\theta_n) \to (\mu,0)$ in $\mathbb{R} \times L_2(0,\pi)$, then $\mu = 6/\pi$.

From this observation, the method of (F') yields that $f_{\lambda_n}\theta_n$ converges to $f\theta$ in $L_1(0,\pi)$, and then the method of (G') may be used to prove that $\theta_n \to \theta$ in $C_0[0,\pi]$. This completes the proof of (i), (ii), (iii).

(iv) By (1.23) and (1.34)

$$c(\mu_n,\theta_n) = \frac{2\sqrt{3g}\, K_{\lambda_n}(1+k_{\lambda_n})}{\lambda_n}\left(\frac{2}{\lambda_n}\int_0^\pi \frac{f_{\lambda_n}(t)\cos\theta_n(t)}{(\frac{1}{\mu_n}+\int_0^t f_{\lambda_n}(w)\sin\theta_n(w)dw)^{1/3}}dt\right)^{-3/2} .$$

From (1.24) it follows that

$$\frac{2\sqrt{3g}\, K_{\lambda_n}(1+k_{\lambda_n})}{\lambda_n} \to \frac{\sqrt{3g}\,\pi}{2h} \tag{2.10}$$

as $n \to \infty$. Now for any $\alpha \in (0,\pi)$,

$$\frac{2}{\lambda_n}\int_0^\pi \frac{f_{\lambda_n}(t)\cos\theta_n(t)}{(\frac{1}{\mu_n}+\int_0^t f_{\lambda_n}(w)\sin\theta_n(w)dw)^{1/3}}dt$$

$$= \frac{2}{\lambda_n}\int_0^\alpha \frac{f_{\lambda_n}(t)\cos\theta_n(t)}{(\frac{1}{\mu_n}+\int_0^t f_{\lambda_n}(w)\sin\theta_n(w)dw)^{1/3}}dt +$$

$$\frac{2}{\lambda_n}\int_\alpha^\pi \frac{f_{\lambda_n}(t)\cos\theta_n(t)}{(\frac{1}{\mu_n}+\int_0^t f_{\lambda_n}(w)\sin\theta_n(w)dw)^{1/3}}dt \quad . \tag{2.11}$$

For any $\varepsilon > 0$, choose $\alpha(\varepsilon) > 0$ such that, for all n sufficiently large $|\cos\theta_n(t) - 1| \le \varepsilon$ for all $t \in [\alpha(\varepsilon),\pi]$. This can be done since $\theta_n \to \theta \in K_0$ uniformly on $[0,\pi]$. Moreover $\alpha(\varepsilon)$ can be chosen so that

$$\left|(\frac{1}{\mu_n}+\int_0^t f_{\lambda_n}(w)\sin\theta_n(w)dw)^{1/3} - (\frac{1}{\mu}+\int_0^\pi f(w)\sin\theta(w)dw)^{1/3}\right| \le \varepsilon$$

for all $t \in [\alpha(\varepsilon),\pi]$, and for all n sufficiently large. From (2.11) it follows that

$$\lim_{n\to\infty} \frac{2}{\lambda_n} \int_0^{\pi} \frac{f_{\lambda_n}(t)\cos\theta_n(t)}{\left(\frac{1}{\mu_n} + \int_0^t f_{\lambda_n}(w)\sin\theta_n(w)dw\right)^{1/3}} dt$$

$$= \lim_{n\to\infty} \frac{2}{\lambda_n} \int_{\alpha(\varepsilon)}^{\pi} \frac{f_{\lambda_n}(t)\cos\theta_n(t)}{\left(\frac{1}{\mu_n} + \int_0^t f_{\lambda_n}(w)\sin\theta_n dw\right)^{1/3}} dt$$

$$\begin{cases} \geq \lim_{n\to\infty} \dfrac{2}{\lambda_n} \displaystyle\int_{\alpha(\varepsilon)}^{\pi} \dfrac{f_{\lambda_n}(t)(1-\varepsilon)}{\left(\frac{1}{\mu_n} + \int_0^{\pi} f_{\lambda_n}(w)\sin\theta_n(w)dw\right)^{1/3}} dt \\ \\ \leq \lim_{n\to\infty} \dfrac{2}{\lambda_n} \displaystyle\int_{\alpha(\varepsilon)}^{\pi} \dfrac{f_{\lambda_n}(t)}{\left(\frac{1}{\mu} + \int_0^{\pi} f(w)\sin\theta(w)dw\right)^{1/3} - \varepsilon} dt \end{cases} \qquad (2.12)$$

However, by (1.24) and (1.25),

$$\lim_{n\to\infty} \frac{2}{\lambda_n} \int_{\alpha(\varepsilon)}^{\pi} f_{\lambda_n}(t)dt = \lim_{n\to\infty} \frac{2}{\lambda_n} \int_0^{\pi} f_{\lambda_n}(t)dt = \frac{\pi}{2h} \qquad (2.13)$$

and

$$\lim_{n\to\infty} \left(\frac{1}{\mu_n} + \int_0^{\pi} f_{\lambda_n}(w)\sin\theta_n(w)dw\right)^{1/3}$$

$$= \left(\frac{1}{\mu} + \int_0^{\pi} f(w)\sin\theta(w)dw\right)^{1/3} . \qquad (2.14)$$

Collecting (2.10) - (2.14), we find that

$$\frac{\pi\sqrt{3g}}{2h} \left(\frac{2h}{\pi}\right)^{3/2} \left[\left(\frac{1}{\mu} + \int_0^{\pi} f(w)\sin\theta(w)dw\right)^{1/3} - \varepsilon\right]^{3/2}$$

$$\leq \lim_{n\to\infty} c(\mu_n,\theta_n)$$

$$\leq \frac{\pi\sqrt{3g}}{2h}\left(\frac{2h}{\pi}\right)^{3/2}(1-\varepsilon)^{-3/2}\left(\frac{1}{\mu}+\int_0^\pi f(w)\sin\theta(w)\,dw\right)^{1/2},$$

and since ε is arbitrary

$$\lim_{n\to\infty} c(\mu_n,\theta_n) = \sqrt{\frac{6gh}{\pi}}\left(\frac{1}{\mu}+\int_0^\pi f(w)\sin\theta(w)\,dw\right)^{1/2}.$$

That this last quantity lies in an interval $[1,M']$ has been established in [1; Theorem 3.9].

(v) An analogous calculation to that just given yields (v).

q.e.d.

COROLLARY 2.6. <u>The statement of this corollary is given in section</u> 1.1.

<u>Proof</u>. By Theorem 2.2, there exists $(\mu_n,\theta_n) \in C_{\lambda_n}$ such that $|\theta|_{C_0[0,\pi]} = \beta$, for any $\beta \in [0,\pi/6)$. The result will follow by the method used in the proof of Theorem 2.3, once it is established that the sequence $\{\mu_n\}$ is bounded. However

$$\theta_n(s) = \frac{2}{3}\int_0^\pi G(s,t)\,\frac{f_{\lambda_n}(t)\sin\theta_n(t)}{\frac{1}{\mu_n}+\int_0^t f_{\lambda_n}(w)\sin\theta_n(w)\,dw}\,dt$$

$$\geq \frac{2}{3}\int_0^\pi G(s,t)\,\frac{f_{\lambda_m}(t)\sin\theta_n(t)}{\frac{1}{\mu_n}+\int_0^t f_{\lambda_m}(w)\sin\theta_n(w)\,dw}\,dt,$$

if $n \geq m$, by Lemma 2.4. Without loss of generality suppose that $\mu_n \to \infty$. Then, as in the proof of Theorem 2.2(iv) it can be shown that there exists $\beta > 0$ such that $\theta_n(s) \geq \beta \sin s$, for all $s \in [0,\pi]$. But this estimate is enough to guarantee (by a routine adaptation of the methods of [1]) that a subsequence $\{\theta_{n(k)}\}$ of $\{\theta_n\}$ converges in $C[\delta,\pi]$ for each $\delta > 0$ to a solution θ of the equation

$$\theta(s) = \frac{2}{3}\int_0^{\pi} G(s,t)\,\frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)\,dw}\,dt \quad .$$

However, we know from [1; Theorem 5.2] that for such a function θ, $\overline{\lim_{s\to 0+}}\,\theta(s) \geq \pi/6$. This contradicts the fact that $|\theta_n|_{C_0[0,\pi]} = \beta < \frac{\pi}{6}$.

q.e.d.

Finally, we have the following result. Let $S = \{(\mu,\theta) \in (0,\infty) \times K_0 :$ (μ,θ) solves (1.47) and $\theta \neq 0\}$. For all $(\mu,\theta) \in S$, the product $f\theta \in L_1(0,\pi)$ ([1; Theorem 4.1]). Let $S' = \{(\mu,\theta) \in S : (\mu,\theta,f\theta)$ is the limit, as $\lambda \to \infty$, in $\mathbb{R} \times C_0[0,\pi] \times L_1(0,\pi)$ of a sequence $(\mu_\lambda,\theta_\lambda,f_\lambda\theta_\lambda)$ where $(\mu_\lambda,\theta_\lambda) \in S_\lambda\}$.

THEOREM 2.7. _If_ C' _is the maximal connected subset of_ S' _which contains_ $(6/\pi,0)$, _then_ C' _is closed, unbounded, and has all the properties attributed to_ C _in_ [1; Theorem 3.9]. _Clearly_ $C' \subset C$.

Proof. This is immediate, since it has been shown that the boundary ∂U of every bounded, open set $U \subset \mathbb{R} \times C_0[0,\pi]$ which contains $(6/\pi,0)$, contains a point of C'. Since the set S' is obviously a closed subset of S, and it has the property that bounded subsets of it are relatively compact, [1; Theorem 3.8], the result is immediate from [1; Theorem A6].

q.e.d.

Remark. Clearly the result of [14], quoted in Theorem 2.2(vii) for periodic waves holds also for solitary waves corresponding to C' or C, if f_λ is replaced by f in the boundary-layer equation.

APPENDIX

Periodic flows of infinite depth

THEOREM. *Suppose that* θ *is an odd, continuous function on* $[-\pi,\pi]$ *with* $0 < \theta(s) < \pi$ *on* $[0,\pi]$, *which satisfies the integral equation*

$$\theta(s) = \frac{1}{6}\int_{-\pi}^{\pi} \frac{1}{\pi} \ell n\left|\frac{\sin((s+t)/2)}{\sin((s-t)/2)}\right| \frac{\sin\theta(t)}{\frac{1}{\mu} + \int_0^t \sin\theta(w)\,dw}\,dt \quad . \tag{A1}$$

Then θ *is analytic on* $[-\pi,\pi]$ *and* $0 < \theta(s) < \pi/2$ *on* $[0,\pi]$. *Moreover if* λ *and* h *are positive real numbers such that*

$$\left(\frac{3g\lambda}{2\pi c^2}\right)^{1/3} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos\theta(t)}{\left(\frac{1}{\mu} + \int_0^t \sin\theta(w)\,dw\right)^{1/3}}\,dt \quad , \tag{A2}$$

then there exists a periodic wave of wavelength λ *on a flow of infinite depth. The velocity of the flow at infinite depth is then* c, *and its velocity at the crest is given by*

$$q_c = \left(\frac{3g\lambda c}{2\pi\mu}\right)^{1/3} \quad ,$$

The free surface may be parametrized by $(x, H^\lambda(x))$, *where*

$$H^\lambda(x) - H^\lambda(0) = -\left(\frac{\lambda^2 c^2}{3g\pi^2}\right)^{1/3} \int_0^{\alpha^{-1}(x)} \frac{\frac{1}{2}\sin\theta(t)}{\left(\frac{1}{2\mu} + \int_0^t \frac{1}{2}\sin\theta(w)\,dw\right)^{1/3}}\,dt \tag{A3}$$

and

$$\alpha(s) = -\left(\frac{\lambda^2 c^2}{3g\pi^2}\right)\int_0^s \frac{\frac{1}{2}\cos\theta(t)}{\left(\frac{1}{2\mu} + \int_0^t \frac{1}{2}\sin\theta(w)\right)^{1/3}}\,dt \quad . \tag{A4}$$

Proof. The proof that θ is analytic and bounded by $\pi/2$ follows exactly as in Theorem 2.2.

As before, there exists a harmonic function $\tilde{\theta}$ on the unit disc such that $\theta(s) = \tilde{\theta}(e^{is})$ for all $s \in (-\pi,\pi]$, and

$$\left.\frac{\partial\tilde{\theta}}{\partial r}\right|_{e^{is}} = \frac{1}{3}\,\frac{\sin\theta(s)}{\frac{1}{\mu} + \int_0^s \sin\theta(w)\,dw} \quad . \tag{A5}$$

Using the expansion of G given in (1.27), it follows from (A1) that for all $s \in (-\pi,\pi]$,

$$\theta(s) = \frac{1}{3}\int_{-\pi}^{\pi}\left\{\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin ks\,\sin kt}{k}\right\}\frac{\sin\theta(t)}{\frac{1}{\mu} + \int_0^t \sin\theta(w)\,dw}\,dt \quad .$$

From this and Fubini's theorem there results that the Fourier series for θ is

$$\sum_{k=1}^{\infty} a_k \sin ks \tag{A6}$$

where

$$a_k = -\frac{1}{3\pi}\int_{-\pi}^{\pi}\cos kt\;\ell n\left(\frac{1}{\mu} + \int_0^t \sin\theta(w)\,dw\right)dt \quad . \tag{A7}$$

It follows that putting

$$\tilde{\tau}(\zeta) + i\tilde{\theta}(\xi) = \sum_{k=1}^{\infty} a_k \xi^k \tag{A8}$$

defines an analytic function on the unit disc, and

$$\tilde{\tau}(e^{is}) + i\tilde{\theta}(e^{is}) = a_0 - \frac{1}{3}\,\ell n\left(\frac{1}{\mu} + \int_0^s \sin\theta(w)\,dw\right) + i\theta(s) \tag{A9}$$

for all $s \in [-\pi,\pi]$, where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{3}\,\ell n\left(\frac{1}{\mu} + \int_0^t \sin\theta(w)\,dw\right)dt$.

Let c and λ be positive real numbers chosen so that (A2) holds. Then an analytic function $\tilde{T} - i\tilde{\Theta}$ can be defined on $R^{\lambda} = \{x + i\eta : -\lambda/2 < x < \lambda/2,\ \eta < 0\}$ by putting

$$\tilde{T}(\zeta) - i\tilde{\Theta}(\zeta) = \tilde{\tau}(\exp(-2\pi i\zeta/\lambda)) + i\tilde{\theta}(\exp(-2\pi i\zeta/\lambda)) \quad .$$

Hence $\Theta(\chi + i0) = -\theta(-2\pi\chi/\lambda)$ and so

$$\left.\frac{\partial\Theta}{\partial\eta}\right|_{\chi+i0} = \frac{2\pi}{3\lambda}\,\frac{-\sin\theta(-2\pi\chi/\lambda)}{\frac{1}{\mu} + \int_0^{-2\pi\chi/\lambda}\sin\theta(w)\,dw}$$

$$= \frac{1}{3}\,\frac{\sin\Theta(\chi)}{\frac{\lambda}{2\pi\mu} + \int_0^{\chi}\sin\Theta(w)\,dw} \quad . \qquad \text{(A10)}$$

Since $|\theta| < \pi/2$ on $[-\pi,\pi]$ it follows by the maximum principle that $|\Theta| < \pi/2$ in R^{λ}.

Now define an analytic function $\tilde{m}$ on R^{λ} by putting

$$\tilde{m}(\zeta) = \int_0^{\zeta} \exp(\tilde{T}(\zeta') - i\tilde{\Theta}(\zeta'))d\zeta' \quad .$$

Since $\Theta(\pm\lambda/2 + i\eta) = 0$ for all $\eta < 0$, and since $|\tilde{T}(\zeta) - i\tilde{\Theta}(\zeta)| \to 0$ as $|\zeta| \to \infty$, $\zeta \in R^{\lambda}$, it follows that $\tilde{m}$ is a conformal mapping from R^{λ} onto an infinite region in the z-plane of the form $S^{\lambda} = \{\chi + iy : -\lambda/2 < x < \lambda/2, y \le H^{\lambda}(x)\}$, and $H^{\lambda\prime}(x) = -\tan\Theta(\tilde{m}^{-1}(x + iH^{\lambda}(x)))$. Since $\tilde{m}$ is invertable, we can define a complex potential $\tilde{\omega} = \tilde{\phi} + i\tilde{\psi}$ on S^{λ} by putting

$$\tilde{\omega}(z) = c\,\tilde{m}^{-1}(z) \quad ,$$

where c was chosen when λ was chosen so that (A2) holds. Then for $z \in S_{\lambda}$,

$$u(z) - iv(z) = -\frac{d\tilde{w}}{dz}$$

$$= -c\exp(-\tilde{T}(m^{-1}(z)))(\cos\tilde{\Theta}(\tilde{m}^{-1}(z)) + i\sin\tilde{\Theta}(\tilde{m}^{-1}(z)))$$

and it follows that $c\exp(-\tilde{T}(m^{-1}(z)))$ is the speed of the flow and $-\Theta(\tilde{m}^{-1}(z))$ is the angle which the negative velocity vector makes with the

x-axis, at a point $z \in S^\lambda$. Moreover $u(z) - i\,v(z) \to -c$ as $|z| \to \infty$, $z \in S_\lambda$. From the definition of $\tilde{\omega}$ it follows that $\tilde{\psi} \to -\infty$ as $|z| \to \infty$, $z \in S^\lambda$, and $\tilde{\psi} = 0$ on the free surface $\Gamma^\lambda = \{(x, H^\lambda(x)) : x \in (-\lambda/2, \lambda/2)\}$.

Finally to show that the free surface condition is satisfied we proceed as follows. By (A8), (A9) and Cauchy's formula there results that

$$1 = \exp(0) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{\exp(\tau(e^{it}) + i\theta(e^{it}))\, i e^{it}}{e^{it}}\, dt$$

$$= \frac{\exp(a_0)}{2\pi} \int_{-\pi}^{\pi} \frac{\cos\theta(t)}{\left(\frac{1}{\mu} + \int_0^t \sin\theta(w)\,dw\right)^{1/3}}\, dt ,$$

and so, by our choice of λ and c,

$$\exp(a_0) = \left(\frac{2\pi c^2}{3g\lambda}\right)^{1/3} . \tag{A11}$$

Hence

$$\tilde{\tau}(e^{is}) = \frac{1}{3}\ln\left(\frac{2\pi c^2}{3g\lambda}\right) - \frac{1}{3}\ln\left(\frac{1}{\mu} + \int_0^s \sin\theta(w)\,dw\right) ,$$

and so

$$\tilde{T}(\chi + i0) = \frac{1}{3}\ln\left(\frac{2\pi c^2}{3g\lambda}\right) - \frac{1}{3}\ln\left(\frac{1}{\mu} + \frac{2\pi}{\lambda}\int_0^\chi \sin\Theta(w)\,dw\right) . \tag{A12}$$

Therefore

$$\frac{d}{d\chi}\left\{\frac{c^2}{2}\exp(-2\tilde{T}(\chi+i0)) + g\,\mathrm{Imag}\,\tilde{m}(\chi+i0)\right\}$$

$$= \frac{2\pi}{3\lambda}\, \frac{c^2\exp(-2\tilde{T}(\chi+i0))\sin\Theta(\chi)}{\frac{1}{\mu} + \frac{2\pi}{\lambda}\int_0^\chi \sin\Theta(w)\,dw} - g\exp(\tilde{T}(\chi+i0))\sin\tilde{\Theta}(\chi+i0)$$

$$= \exp(\tilde{T}(\lambda+i0))\{\frac{2\pi c^2}{3\lambda}\frac{3g\lambda}{2\pi c^2}\sin\theta(\lambda) - g\sin\theta(\lambda)\}$$

$$= 0 \quad .$$

To complete the proof of the theorem we must verify that (A3), (A4) give the wave profile. This is a routine calculation based on the method used in the proof of Theorem 1.3.

q.e.d.

Though the proof of this last theorem is in many respects similar to that of Theorem 1.3, we have included it in order to obtain the following corollary. We need the notion of a conjugate function which is defined as follows. If u is an L_2-function whose Fourier series is $a_0 + \sum_{k=1}^{\infty} (a_k \cos ks + b_k \sin ks)$, then the function conjugate to u is denoted by $\underset{\sim}{C}u$ and is the L_2-function whose Fourier series is $\sum_{k=1}^{\infty} (a_k \sin ks - b_k \cos ks)$ [2].

COROLLARY. _If_ θ _satisfies_ (A1) _for some_ $\mu > 0$, _then_ θ _satisfies the equation_

$$\theta(s) = \frac{\nu}{6}\int_{-\pi}^{\pi} \frac{1}{\pi}\,\ell n\left|\frac{\sin((s+t)/2)}{\sin((s-t)/2)}\right| \exp(-3\underset{\sim}{C}\theta(t)\sin\theta(t)dt$$

where $\nu = \dfrac{3g\lambda}{2\pi c^2}$.

Proof. By (A6) - (A8)

$$-3\underset{\sim}{C}\theta(t) = 3\tilde{\tau}(e^{it})$$

$$= 3a_0 - \ell n(\frac{1}{\mu} + \int_0^t \sin\theta(w)dw) \quad ,$$

whence by (A11)

$$\exp(-3\underset{\sim}{C}\theta(t)) = \frac{2\pi c^2}{3g\lambda}\left(\frac{1}{\frac{1}{\mu} + \int_0^t \sin\theta(w)\,dw}\right) .$$

Substituting ıs last expression into (A1) gives the required result.

q.e.d.

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It is shown that large amplitude solitary water waves arise as the limit of periodic waves whose wavelength increases indefinitely. This result is obtained after a new version of the Nekrasov integral equation for periodic waves has been derived. Its resemblance to the equation for solitary waves [1] leads to this convergence result once the global existence proof for solitary waves given in [1] has been taken into account.

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